Solving under-constrained numerical constraint satisfaction problems with IBEX

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A numerical constraint satisfaction problem (NCSP) consists in finding the variables values $x \in \mathbb{R}^n$ in a given box domain $[x] \in \mathbb{IR}^n$ satisfying equality constraints $h(x) = 0$, with $h : \mathbb{R}^n \to \mathbb{R}^{n_e}$ where n_e is the number of equality constraints, and inequality constraints $g(x) \leq 0$, with $g : \mathbb{R}^n \to \mathbb{R}^{n_i}$ where n_i is the number of inequality constraints. The solution set to be computed is denoted $\Sigma := \{x \in [x] : h(x) = 0, g(x) \leq 0\}$. Computing exactly Σ is impossible in general, and numerical constraint solvers usually compute a paving $\mathcal{P} \subseteq \mathbb{IR}^n$, i.e., a finite set of boxes, consisting of *inner* boxes and *unknown* boxes, i.e., $\mathcal{P} = \mathcal{I} \cup \mathcal{U}$ and $\mathcal{I} \cap \mathcal{U} = \emptyset$. Inner boxes are proved to contain solutions, but the exact interpretation of these boxes actually depends on the structure of the problem. No information is available for unknown boxes, i.e., they may or may not contain solutions. Still, NCSP solvers are complete and lose no solution therefore $\Sigma \subseteq \cup\mathcal{P}$.

Constraint solvers handle efficiently NCSPs where there is no equality constraint ($n_e = 0$), in which case an inner box contains only solutions (see Figure 1-right), and well constrained systems of equations ($n_e = n$), in which case an inner box contains only one solution (see Figure 1-left). This twofold interpretation of inner boxes with respect to the problem structure seems inhomogeneous. Intermediate cases with less equations than variables, where Σ is a manifold of dimension $n - n_e \in \{1, \ldots, n-1\}$, have been the topic of several research but were not included in the general framework of NCSPs. In this case, constraint solvers typically output a large number of unknown boxes covering the solution set.

We propose a generalized interpretation of inner boxes: A box $[x]$ is called inner if we can choose $n - n_e$ coordinates x_P with $P \subseteq N := \{1, \ldots, n\}$, called parameters, such that for each $x_P \in [x_P]$ there exists a unique choice of the other n_e coordinates $x_{N\backslash P} \in [x_{N\backslash P}]$, such that x is a solution. This interpretation subsumes the usual one. Indeed, when $n_e = 0$ we obtain $\forall x_N \in [x_N], \exists ! x_\emptyset \in [x_\emptyset], x \in \Sigma$, which is simplified as $\forall x \in [x], x \in \Sigma$; Also, when $n_e = n$ we obtain $\forall x_\emptyset \in [x_\emptyset], \exists ! x_N \in [x_N], x \in \Sigma$, which simplifies to $\exists ! x \in [x], x \in \Sigma$. Finally, a solution set of dimension $n - n_e \in (0, n)$ basically traverses the inner box "in parallel" to the subspace x_P (see Figure 1-center).

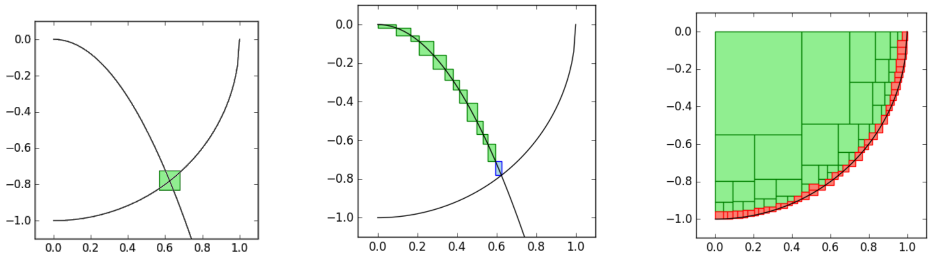

Figure 1: From left to right: Zero, one and full dimensional solution sets. Inner boxes are shown in green and unknown boxes are shown in red. The unknown box of the zero dimensional NCSP contains a singular solution. Dark and light green inner boxes of the one dimensional NCSP correspond to $P = \{1\}$ and $P = \{2\}$ respectively.

Classically, proving that a box $[x]$ is inner for full dimensional NCSPs (i.e., $n_e = 0$) is done by checking that its inequalities interval evaluation $[g]([x])$ is nonpositive; proving that a box is inner for zero-dimensional NCSPs (i.e., $n_e = n$) is done by using the interval Newton operator. For positive but not full dimensional NCSPs, we use a parametric interval Newton operator, whose success exactly matches the semantic of inner boxes. Parameters are chosen dynamically by studying the Jacobian of the equality constraints in order to determine the direction of the solution set (typically by applying a LU-decomposition with pivoting to discover a square sub-matrix with good conditioning). It is also critical to allow the interval Newton to ”inflate” the current box, no more considering its intersection with the previous one: Indeed, the splitting step of the branch-and-bound may create solutions on the boundary of variable domains that cannot be selected as parameters, and which therefore require an inflation for the parametric interval Newton to succeed. This solving process, called IBEXSOLVE, has been implemented in the IBEX library and can be downloaded at `http://www.ibex-lib.org/`. Several case studies will be presented from geometry, robotics and phase diagrams computation.

Finally, handling carefully the interaction between manifolds defined by equality constraints and inequality constraints, i.e., manifolds with boundaries, is nontrivial. The current theory and implementation allows building an atlas of a manifold without boundary, each inner box giving rise to a chart of the atlas. But, in the presence of a boundary, a so-called boundary box is identified, but not yet proven to be a chart.