A 3-TERM OPTIMIZATION CRITERION FOR FASTER INVERSION IN MICROWAVE TOMOGRAPHY

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ABSTRACT

The problem of microwave tomography consists of finding the electrical characteristics (permittivity and conductivity) of a medium using scattered fields measured under various conditions of illumination. Such inversion can be achieved with a variety of techniques based upon the minimization of an appropriate criterion. The popular current source inversion (CSI) technique is one of those. In this paper, the CSI technique is presented and its main limitations are underlined. Then, in order to overcome these limitations, a generalized form of CSI is developed. An analysis of the time consuming operations of this new algorithm is done, and a new technique is proposed; it proves to be faster than the other two methods, the speed increase ranging between factors of 2 and 5. Performance of the algorithms is illustrated with examples based on synthetic data.

Index Terms— Microwave Tomography, Non-linear Inversion, Contrast Source Inversion

1. INTRODUCTION

In a microwave tomography (MWT) experiment, an object under test (OUT) is illuminated under a variety of conditions and for each of them, the field scattered by the OUT is obtained at a set of measurement points. Using these measurements, the objective is to reconstruct the conductivity and permittivity distributions within the object.

MWT may prove to be particularly interesting in various applications such as non destructive testing and biomedical imaging. In the latter case, two of the most interesting advantages of MWT are the large sensitivity to the physiological state of a patient and the high permittivity contrast between pathological and healthy tissues which, in many cases, is much higher than the transmittance contrast of X-rays (e.g., for cancerous breast tissues, the permittivity contrast varies by a factor of 5 in MWT while the variation is of about 10% for X-rays) [1].

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In spite of such advantages, MWT uses long wavelengths compared to the structural features of the OUT. Consequently, usual ray propagation techniques are not suitable. Instead, an integral equation formulation, which is highly non-linear and *ill-posed*, must be used [2]. This significantly complicates the resolution of the *inverse problem*.

In view of these characteristics, two types of techniques have been proposed in literature. In the first one, the tomography equations are linearized in some way [3, 4]. The resulting algorithms are fast but lack accuracy for high contrast scatterers, which makes them hardly suitable for biomedical applications. The other types of algorithms solve the nonlinear equations [4, 5]. They yield better solutions, but they may present a high computation cost, which is a critical issue for reconstruction of OUTs of realistic sizes.

The widespread CSI technique belongs the latter category [4, 6]. The inversion is done by the minimization of a 2-term criterion. It yields good results for high contrast OUTs and the computational burden remains acceptable because CSI does not rely exact resolutions of the *direct problem*. However, this method presents limitations that will be underlined below. In order to overcome them, we propose a new technique, referred to as 2TC, that can be viewed as a generalization of CSI. While proving faster than CSI, further improvements are desirable for actual use in real life applications. Therefore, after analysis of the time consuming stages of this method, another technique, based on minimization of a three-term criterion, is derived. Is efficiency and performance are illustrated by results obtained with synthetic data.

2. CONTEXT

By using the theorem of *equivalent volumes* and the method of moments (MoM) [7], we can derive the two discretized equations of MWT [4]:

$$y_i = G_o X E_i \tag{1a}$$

$$E_i = E_i^0 + G_c X E_i \tag{1b}$$

where y is the measured field vector. Matrices G_o and G_c represent discretized Green functions. Vector E and E^0 respectively denote the total electrical field with and without the OUT. $X = \mathrm{diag}\{x\}$ is a diagonal matrix whose elements are the contrast values of the OUT. Index i refers to the i-th emitter. The goal of MWT is to find the value of x. However, we note that the total field E_i is also an unknown of the problem.

By substituting (1b) into (1a), we obtain:

$$y_i = G_o(I - G_cX)^{-1}E_i^0 \tag{2}$$

which illustrates the non-linearity of the problem. One possible way of inverting the nonlinear problem is to solve (2) directly [8]. This is quite complex due to the form of the equation. Another option is to solve (1a) under the constraint (1b). This is done in [9] but is very time consuming. The simplest way, from a calculation point of view, and probably the most popular, is to minimize a criterion composed of the weighted sum of two terms: (*i*) the square norm of the error on (1a) and (*ii*) the square norm of the error on (1b) [10]. Note that this approach can be implemented in several different manners based upon equivalent representations of system (1). For example, from the *equivalent volume* theorem, polarization currents w_i can be computed as:

$$w_i = XE_i \tag{3}$$

which yields the following equivalent form of (1):

$$y_i = G_ow_i \tag{4a}$$
$$w_i = XE_i^0 + XG_cw_i \tag{4b}$$

3. TWO-TERM ALGORITHMS

3.1. CSI method

The widespread CSI method solves the MWT problem by minimizing a two-term criterion based on (4). More precisely, the CSI criterion F_{CSI} is defined by:

$$F_{\mathrm{CSI}} = \frac{\sum_i \|y_i - G_ow_i\|^2}{\sum_i \|y_i\|^2} + \frac{\sum_i \|XE_i^0 - w_i + XG_cw_i\|^2}{\sum_i \|XE_i^0\|^2} \tag{5}$$

F_{CSI} is minimized by alternately updating w and x, thereby taking advantage of the quadratic structure of the criterion with respect to (w.r.t.) each the unknown when the other one is fixed. Each minimization step is performed through a single iteration of a conjugate gradient (CG) algorithm. However, it can be shown that, due to the presence of unknown x in the denominator of the second term of the criterion, minimization of F_{CSI} can lead to a degenerate solution (i.e., minima reached for infinite values of components of x). In addition, the denominator of the second term of F_{CSI} (which depends on x) is not accounted for in gradient computations, thereby raising doubts about the convergence properties of the method.

3.2. 2TC method

We now present a new inversion technique based on a two-term criterion, referred to as 2TC, which can be viewed as a generalization of CSI. In order to avoid possible degenerate solutions, the following simplified criterion is used:

$$F_{\mathrm{2TC}} = \sum_i \|y_i - G_ow_i\|^2 + \lambda \sum_i \|XE_i^0 - w_i + XG_cw_i\|^2 \tag{6}$$

where λ is a weighting factor that has to be set by hand. Selection of the value of parameter λ is similar to heuristic specification of a regularization parameter commonly used when solving *ill-posed inverse problems*.

Efficient minimization of F_{2TC} is carried out by solving (6) w.r.t. w and x in an alternate manner. This allows us to take advantage of the quadratic nature of F_{2TC} w.r.t. w and x, respectively. The update formulas for exact minimizer on w and x readily take the following form:

$$(G_o'G_o + \lambda A'A)w_i = G_o'y_i - \lambda A'XE_i^0 \tag{7}$$
$$A = XG_c - I$$
$$\sum_i \mathbf{\Delta}_{w_i}'\mathbf{\Delta}_{w_i}x = \sum_i \mathbf{\Delta}_{w_i}'w_i \tag{8}$$
$$\mathbf{\Delta}_{w_i} \stackrel{\triangle}{=} E_i^0 + G_cw_i$$

where I is the identity matrix and the $\cdot'$ represents the transposed conjugate operation. From a practical standpoint, solving (8) does not present any computational difficulty, thanks to the diagonal structure of $\mathbf{\Delta}_{w_i}$. However, the normal matrix in the left hand side of (7) depends on variable x; this implies that solution of (7) requires the inversion of a full matrix *at every iteration*, which is intractable in practice. In order to overcome this difficulty, we propose to solve (7) approximately through a limited number of CG iterations. As shown below the resulting procedure compares favorably with CSI. It nonetheless presents the disadvantage of being based on two intertwined iterative procedures, which limits its computational efficiency. We now derive a procedure based on a three-term criterion which further reduces the computational load.

4. THREE-TERM ALGORITHM

Our goal is to improve the convergence speed of the method by eliminating the need for intertwined iterative procedures. This goal can be achieved by solving a set of equations equivalent to (1), but whose matrices do not depend on unknown quantities. Such a set of equations can be easily derived from (1) and (3). This yields the following equivalent system of three equations:

$$y_i = G_ow_i \tag{9a}$$
$$E_i = E_i^0 + G_cw_i \tag{9b}$$
$$w_i = XE_i \tag{9c}$$

Using the same approach as in section 3.2, we infer that the solution to (9) can be obtained through minimization of the following three-term criterion (3TC):

$$F_{\mathrm{3TC}} = \sum_i \|y_i - G_o w_i\|^2 + \lambda_1 \sum_i \|E_i^0 - E_i + G_c w_i\|^2 + \lambda_2 \sum_i \|w_i - X E_i\|^2 \qquad (10)$$

where λ_1 and λ_2 are two weighting factors. It should be underlined that minimization of F_{2TC} and F_{3TC} may not yield the same solution: while (9c) is exactly satisfied by construction in the 2TC method, it is only approximately satisfied by the (3TC) technique. However, results will show that approximate fulfillment of constraints is sufficient for obtaining satisfactory results. It should also be mentioned that the 3TC method involves three sets of unknowns instead of two, which all have to be estimated in an efficient manner.

As before, F_{3TC} is minimized iteratively through alternate determination of one set of variables while the other two remain fixed. Minimization of F_{3TC} w.r.t. w yields:

$$(G_o' G_o + \lambda_1 G_c' G_c + \lambda_2 I)\, w_i = G_o' y_i - \lambda_1 G_c' (E_i^0 - E_i) + \lambda_2 X E_i \quad (11)$$

and we note that the normal matrix is independent from E and x and can be determined solely from the physical characteristics of the tomograph and of the weighting factors λ_1 and λ_2. Thus, the normal matrix needs to be inverted once for the whole procedure.

Similarly, the update equations for E and x take the following form:

$$(\lambda_1 I + \lambda_2 X' X)\, E_i = \lambda_1 (E_i^0 + G_c w_i) + \lambda_2 X' w_i \qquad (12)$$

$$\sum_i \mathrm{diag}\{E_i'\}\, x = \sum_i \mathrm{diag}\{E_i'\}\, w_i \qquad (13)$$

The normal matrices in (12) and (13) present a diagonal structure, thereby greatly simplifying the updates of E and x. Therefore, the 3TC method meets the goal of eliminating intertwined iterations an the simple update formulas indicate that a significant reduction in computation time could be achieved with respect to CSI ans 2TC.

For comparison purposes, the calculation cost for one iteration of the 3TC method is $O(3n^2)$, while it is $O(2Kn^2)$ for the 2TC method, where n is the number of unknowns and K the number of GC steps needed to solve (7).

5. REGULARIZATION

In this section we very briefly introduce the concept of regularization. As indicated in the introduction, MWT is an *ill-posed* problem with the consequence of a high sensitivity of the solutions to measurement noise. *Regularization* is a well known approach to alleviating the problem. *Regularization* is commonly introduced through addition of a penalty term to the estimation criterion. Here, we will only consider penalty terms of the form:

$$\lambda_{reg} \|Dx\|^2 \qquad (14)$$

where parameter λ_{reg} weight the importance of the penalty term with respect to the other components of the estimation criterion. Matrix D is chosen so as to favor desirable properties of the solution (e.g., smoothness) while lending itself to easy insertion into the update formula of x. In the sequel, methods will be compared with and without *regularization*.

6. RESULTS

In this section, we compare results obtained by the CSI, the 2TC and 3TC methods. This will highlight the efficiency of the latter in the regularized case.

We use a 2D example. The unknown domain is square with one wavelength side. The scatterer is formed by two concentric square cylinders of value $1 - 0.5j$ and $0.5 - j$. We use a 32×32 discretization, which yields 1024 unknowns. There are 32 antennas, and each can act either as an emitter or a transmitter. Data is obtained by simulation and a Gaussian white noise is added such that the signal to noise ratio (SNR) is equal to 20dB. Two performance criteria are considered: the quality of the solution and the computing time. The first one is measured by the mean square error according to $\mathrm{RMSE} = \|x - x_o\|^2 / \|x_o\|^2$, where x_o and x are the true and reconstructed contrasts, respectively.

For all methods, the same stopping criterion on the iterations is a threshold test on the norm of the gradient. For the three unregularized methods, we have checked that such a stopping criterion provides solutions that are nearly as good as possible in terms of RMSE. On the other hand, the tuning parameters of 2TC and 3TC are set so that their solutions are as close as possible from that of CSI in terms of RMSE.

	Method	RMSE	Time(s)
Without Regularization	CSI	0.1472	165
	2TC	0.1457	107
	3TC	0.4609	53
With regularization	CSI	0.1277	125
	2TC	0.1269	109
	3TC	0.1231	32

Table 1. Comparison between the three methods

On the one hand, the unregularized form of 2TC provides a solution comparable to that of CSI, in significantly less time. As expected, a coherent approach to the minimization of a well-designed fidelity-to-data criterion yields a more efficient method.

On the other hand, the unregularized form of 3TC is competitive in terms of computing time, but not in terms of RMSE.

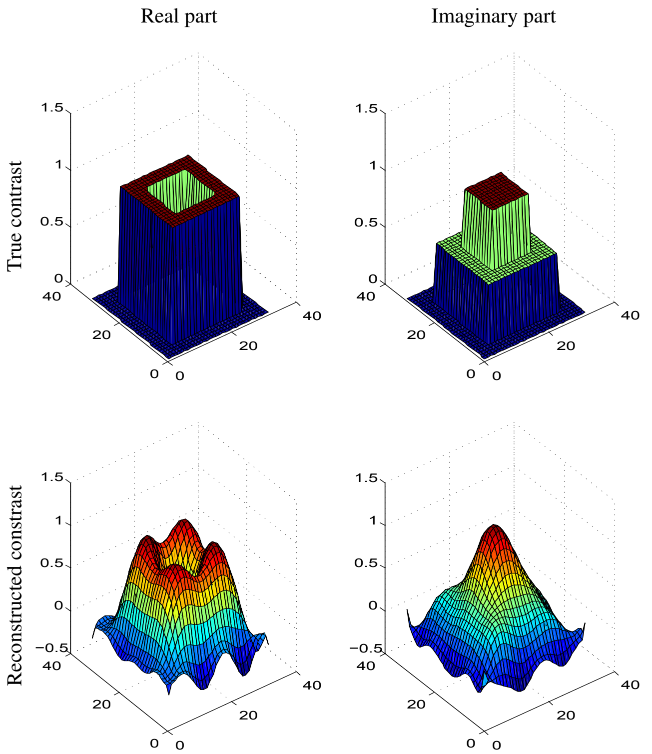

Fig. 1. Real and imaginary parts of true contrast and regularized 3TC solution.

According to our experience, the fact that neither (9b) nor (9c) perfectly hold jeopardizes the behavior of the unregularized solution. Fortunately, regularization allows to strongly enhance the RMSE of the 3TC solution. While all regularized methods yield faster convergence or better solutions than their unregularized versions, the 3TC solution is by far the fastest to compute, while being comparable in terms of RMSE. It is depicted in Figure 1. The global shape and the values of the contrast are recovered. The reconstructed solution is quite comparable to a low-resolution version of the true solution, which is not surprising given that MWT uses long wavelengths compared to the sharp variations of the true object.

7. CONCLUSION

The present paper is devoted to the specification of new methods for the resolution of MWT problems. Firstly, we analyzed the well known technique of CSI. To make for its deficiencies, we proposed a variation of it, which is also based on the minimization of a 2-term criterion. This 2TC method is faster than CSI, but it does not achieve an actual breakthrough in terms of computing time. Therefore, we proposed a deeper modification of the criterion to minimize. Block coordinate-wise minimization of our new 3-term criterion only involves simple operations. Although the efficiency of this 3TC technique is disappointing when regularization is implicitly enforced by early stopping of the iterative minimization scheme, it becomes highly competitive when a proper penalization term is added to the minimized criterion.

8. REFERENCES

[1] S. Y. Semenov, R. H. Svenson, A. E. Boulyshev, A. E. Souvorov, V. Y. Borisov, Y. Sizov, A. N. Starostin, K. R. Dezern, G. P. Tatsis, and V. Y. Baranov, "Microwave tomography: Two-dimensional system for biological imaging", *IEEE Trans. Biomed. Eng.*, vol. 43, no. 9, pp. 13–26, 1996.

[2] A. Franchois and C. Pichot, "Microwave imaging – complex permittivity reconstruction with a Levenberg-Marquardt method", *IEEE Trans. Ant. Propag.*, vol. 45, no. 2, pp. 203–215, 1997.

[3] A. Abubakar and P. M. van den Berg, "A robust iterative method for Born inversion", *IEEE Trans. Geosci. Remote Sensing*, vol. 42, no. 2, pp. 342–354, 2004.

[4] P. van den Berg and A. Abubakar, "Inverse scattering and its applications to medical imaging and subsurface sensing", *Radio Sci. Bull.*, , no. 303, pp. 869–877, 2002.

[5] A. Massa, D. Franceschini, G. Franceschini, M. Pastorino, M. Raffetto, and M. Donelli, "Parallel GA-based approach for microwave imaging applications", *IEEE Trans. Ant. Propag.*, vol. 53, no. 10, pp. 3118–3126, 2005.

[6] P. M. van den Berg and R. E. Kleinman, "A contrast source inversion method", *Inverse Problems*, vol. 13, pp. 1607–1619, 1997.

[7] C. A. Balanis, *Advanced Engineering Electromagnetics*, Wiley, 1989.

[8] H. Carfantan, A. Mohammad-Djafari, and J. Idier, "A single site update algorithm for nonlinear diffraction tomography", in *Proc. IEEE ICASSP*, Munich, Germany, Apr. 1997, pp. 2837–2840.

[9] N. Joachimowicz, C. Pichot, and J.-P. Hugonin, "Inverse scattering: An iterative numerical method for electromagnetic imaging", *IEEE Trans. Ant. Propag.*, vol. 39, no. 12, pp. 1742–1752, 1991.

[10] M. Donelli and A. Massa, "Computational approach based on a particle swarm optimizer for microwave imaging of two-dimensional dielectric scatterers", *IEEE Trans. Microwave Theory Tech.*, vol. 53, no. 5, pp. 1761–1775, 2005.