Abstract—This paper investigates the optimal-cost reachability problem in the context of time Petri nets, where a rate cost is associated with each place. This problem consists in deciding whether or not there exists a sequence of transitions reaching, with minimal cost, a given goal marking. This paper shows that for some subclasses of cTPNs, the optimal-cost reachability problem can be solved more efficiently using a method based on the state classes, without resorting to linear programming or splitting state classes.

I. INTRODUCTION

Time Petri nets (TPNs for short) are a simple yet powerful formalism useful to model and verify real-time, concurrent systems that are therefore subject to time constraints. In TPNs, a firing interval, associated with each transition, specifies the minimum and maximum duration it must be maintained enabled before its firing. Thus, TPNs can model time constraints, even when the exact delays or durations of events are not known. The verification of a TPN is based on the state space abstraction that takes into account the time constraints of the model, while preserving its markings and firing sequences.

This paper deals with the cost time Petri nets (cTPNs for short) and investigates the optimal cost reachability problem. A cTPN is a TPN extended with rate costs associated with its places. The rate cost of a place \( p \) is the sojourn cost (per time unit) of each token in place \( p \). These rate costs do not affect the behaviour of the TPN but they allow to determine the sojourn cost in each marking and also the cost of firing a sequence of transitions.

The optimal cost reachability problem can be stated as the problem of deciding if there exists a sequence of transitions \( \omega \) that allows to reach with minimal cost a given goal marking. Starting from the initial marking, the marking of the model evolves by firing transitions. Each time a transition is fired, some tokens are consumed and some others are produced. We define, for each transition \( t \), a rate cost called incidence rate cost of \( t \) as the sum of rate costs of tokens produced by \( t \) minus the sum of rate costs of tokens consumed by \( t \). We show that for sequences such that the incidence rate costs of their transitions are all non-negative or all non-positive, their optimal-costs can be computed more efficiently based on the state class method without using techniques of linear programming or decomposing state classes, as done previously. Moreover, we show how to compute the optimal-cost of sequences such that the firing interval of their transitions are all singular. Therefore, the optimal cost reachability problem can be solved more efficiently for some subclasses of cTPNs.

Such subclasses might seem restrictive but can in fact model a wide range of applications. Consider for instance a leak in a pressure pipe: until its fixing, the rate at which the water leaks will surely increase, as the leak keeps getting larger. The subclass of model can also describe any economic system based on rarefying resources such as oil or Bitcoins, where the cost of things keep increasing.

The optimal-cost reachability problem has been addressed for Priced Timed Automata (PTAs for short) in [1]–[5] using the region graphs and the zone based graphs. In [1], the authors have proved the decidability of the optimal-cost problem for PTAs with non-negative costs. In [2]–[4], the computation of the optimal-cost to reach a goal location is based on a forward exploration of zones extended with linear cost functions. The linear cost function of a zone gives the optimal-costs to reach each state within the zone. In [5], the authors have improved the approach, developed in [2]–[4], so as to ensure termination of the forward exploration algorithm, even when clocks are not bounded and costs are negative, provided that the PTA is free of negative cost cycles.

For priced timed/time Petri nets, the optimal-cost reachability problem has been addressed in [6], [7]. In [6], the considered model is a timed arc Petri net, under weak firing semantics, extended with rate costs and firing costs associated with places and transitions, respectively. The computation of the optimal-cost for reaching a goal marking is based on similar techniques to those of PTAs [1]. In [7], the authors have investigated the optimal-cost reachability problem for time Petri nets where each transition has a firing cost and each marking has a rate cost (represented as a linear rate cost function over markings). To compute the optimal-cost to reach a goal marking, the authors have first revisited, to include costs, the state class graph method and then reduced the computation, as all other techniques, to a linear programming problem.

The rest of the paper is organised as follows. Section II is devoted to the TPN model, its semantics and its state class graph method. Section III presents the TPN extended with
costs considered here and then defines the cost of a run and the optimal-cost of a sequence. It also shows how to rewrite the cost of a run based on the incidence rate costs of its transitions. Section IV investigates efficient computation procedures of the optimal-cost of firing a sequence of transitions from a state class that need neither minimisation techniques nor splitting state classes. Section V shows by means of a case study how the optimal-costs are computed. Section VI concludes the paper by some future work.

II. TIME PETRI NETS

A. Definition and semantics

Syntactically, a time Petri net is a Petri net where a firing time intervals is associated with each transition.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ be the set of non-negative integers, the set of integers, the set of non-negative rational numbers and the set of non-negative real numbers, respectively. Let $\mathbb{Q}^+_{\leq}$ be the set of non-empty intervals of $\mathbb{R}^+$ whose bounds are in $\mathbb{Q}^+ \cup \{\infty\}$, respectively. For an interval $I \in \mathbb{Q}^+_{\leq}$, $\downarrow I$ and $\uparrow I$ denote its lower and upper bounds, respectively.

Formally, a TPN is a tuple $\mathcal{N} = (P, T, pre, post, M_0, Is)$ where $P$ and $T = \{t_1, ..., t_m\}$ (with $m > 0$) are finite sets of places and transitions such that $P \cap T = \emptyset$, $pre$ and $post$ are the backward and the forward incidence functions ($pre: P \times T \rightarrow \mathbb{N}$), $M_0$ is the initial marking ($M_0: P \rightarrow \mathbb{N}$), and $Is$ is the static firing interval function ($Is: T \rightarrow \mathbb{Q}^+_{\leq}$).

Let $\mathcal{N}' = (P, T, pre, post, M_0, Is)$ be a TPN, $M: P \rightarrow \mathbb{N}$ a marking and $t_i$ a transition of $T$. Transition $t_i$ is enabled for $M$ iff all required tokens for firing $t_i$ are present in $M$, i.e., $\forall p \in P, M(p) \geq pre(p, t_i)$.

In this paper, we use the original semantics of the TPN [8]: If a transition is multi-enabled in some state, only one instance of this transition is considered (single-server semantics), and when a transition is fired, all transitions disabled and enabled again, during this firing, are newly enabled.

We denote $En(M)$ the set of all transitions enabled for $M$, i.e., $En(M) = \{t_i \in T \mid \forall p \in P, pre(p, t_i) \leq M(p)\}$.

If $M$ results from firing some transition $t_f$ from some marking, $Nw(M, t_f)$ denotes the set of all transitions newly enabled in $M$, i.e., $Nw(M, t_f) = \{t_i \in En(M) \mid t_i = t_f \lor \exists p \in P, M(p) - Post(p, t_f) < pre(p, t_i)\}$.

The TPN state is defined as a pair $s = (M, I)$, where $M$ is a marking and $I$ is a firing interval function ($I: En(M) \rightarrow \mathbb{Q}^+_{\leq}$). The initial state of the TPN model is $s_0 = (M_0, I_0)$ where $I_0(t_i) = Is(t_i)$, for all $t_i \in En(M_0)$. The TPN state evolves either by elapsing time or by firing transitions. When a transition $t_i$ becomes enabled, its firing interval is set to its static firing interval $Is(t_i)$. The bounds of this interval decrease synchronously with time, until $t_i$ is fired or disabled by another firing. $t_i$ can fire if the lower bound of its firing interval reaches 0 but must fire, without any additional delay, as far as any conflict avoids it, if the upper bound of its firing interval reaches 0. The firing of a transition takes no time and leads to a new marking.

Let $(M, I)$ and $(M', I')$ be two interval states of the TPN model, $\theta \in \mathbb{R}^+$ and $t_f \in T$. We write $(M, I) \xrightarrow{\theta} (M', I')$, also denoted $(M, I) + \theta$, iff from state $(M, I)$, we reach the state $(M', I')$ by a time progression of $\theta$ units, i.e.,

$$\forall t_i \in En(M), \theta \leq \uparrow I(t_i), M' = M$$

$$\forall t_j \in En(M'), I'(t_j) = [\text{Max}(\downarrow I(t_j) - \theta, 0), \uparrow I(t_j) - \theta]$$

We write $(M, I) \xrightarrow{t_f} (M', I')$ iff from state $(M, I)$, we reach the state $(M', I')$ by firing immediately the transition $t_f$, i.e.,

$$t_f \in En(M), \downarrow I(t_f) = 0$$

$$\forall p \in P, M'(p) = M(p) - pre(p, t_f) + post(p, t_f)$$

$$\forall t_i \in En(M'), I'(t_i) = \begin{cases} Is(t_i) & \text{if } t_i \in Nw(M', t_f) \\ I(t_i) & \text{otherwise} \end{cases}$$

We also use the abbreviation $(M, I) \xrightarrow{\theta t_f} (M', I')$ for $(M, I) \xrightarrow{\theta} (M', I)$.

The TPN state space is the transition system $(S, \rightarrow, s_0)$, where $s_0$ is the initial state of the TPN and $S = \{s_i \mid s_0 \xrightarrow{\ast} s\}$ ($\xrightarrow{\ast}$ being the reflexive and transitive closure of the relation $\rightarrow$ defined above) is the set of reachable states of the model.

A run in the TPN state space $(S, \rightarrow, s_0)$, starting from a state $s_1$, is a sequence $s = s_1 \xrightarrow{\theta_1} s_2 \xrightarrow{\theta_2} s_3 \ldots$. Sequences $\theta_1 \theta_2 \theta_3 \ldots$ and $t_1 t_2 \ldots$ are the timed trace and the trace (firing sequence) of $s$, respectively. A marking $M$ is reachable iff $\exists s \in S$ s.t. its marking is $M$. The runs of a TPN are all the maximal runs starting from its initial state $s_0$.

B. State class graphs

Among the TPN state space abstractions proposed in the literature, we consider here the state class graph (SCG) [9], [10]. A SCG state class $\alpha$ consists of a marking $M$ and a conjunction $F$ of atomic constraints$^1$ over the firing dates of the enabled transitions in marking $M$ and the firing date, denoted by $L_F$ of the transition leading to $\alpha$. It represents an over-approximation of the set of states reached by the same firing sequence from the initial TPN state. Note that for convenience purposes, firing delays in the classical SCG state classes in [9] are replaced by firing dates. The formula $F$ characterises the union of the firing date domains of all states within $\alpha$, reached by the same firing sequence from the initial state of the TPN.

The initial SCG state class of the TPN is the pair $\alpha_0 = (M_0, F_0)$, where $M_0$ is the initial marking and $F_0 = \bigwedge_{t_i \in En(M_0)} \downarrow Is(t_i) \leq L_t - L_0 \leq \uparrow Is(t_i)$, where $L_t$ is a non-negative real valued variable representing the firing date of the transition $t_i$ and $L_0$ is a variable representing the date of $\alpha_0$, which is supposed to be 0 for the initial state class.

From the practical point of view, $F$ is represented by a Difference Bound Matrix (DBM in short) [11]. The DBM of $F$ is a square matrix $D$, indexed by variables of $F$. Each entry

$^1$An atomic constraint is of the form $x - y < c$, where $x$, $y$ are real valued variables, $c \in \mathbb{Q} \cup \{\infty\}$ and $\mathbb{Q}$ is the set of rational numbers (for economy of notation, we use operator $\leq$ even if $c = \infty$).
$d_{ij}$ represents the atomic constraint $L_i - L_j \leq d_{ij}$. If there is no upper bound on $L_i - L_j$ with $i \neq j$, $d_{ij}$ is set to $\infty$. Entry $d_{ii}$ is set to 0. Although the same non-empty domain may be encoded by different DBMs, they have a canonical form. The canonical form of a DBM is the representation with tightest bounds on all differences between variables, computed by propagating the effect of each entry through the DBM. A DBM can be seen as the matrix representation of a graph, called a constraint graph [12]. Its canonical form can be computed in $O(n^3)$, $n$ being the number of variables in the DBM, using a shortest path algorithm, like Floyd-Warshall’s all-pairs shortest path algorithm [13].

Let $C_S$ be the set of all syntactically correct SCG state classes and $succ$ a state class successor function: $C_S \times T \rightarrow C_S \cup \{\emptyset\}$, defined by: $\forall \alpha = (M, F) \in C_S, \forall t_f \in T$,

- $succ(\alpha, t_f) \neq \emptyset$ iff $t_f \in En(M)$ and the following formula is consistent (i.e., satisfiable): $F \land (\bigwedge_{t \in En(M)} L_f \leq L) \land \bigwedge_{t \in Nw(M, t_f)} \downarrow Is(t) \leq L - L_j \leq \uparrow Is(t)$. Notice that without loss of generality, for economy of notations, we suppose that the transitions of $En(M)$ are different from those newly enabled by transition $t_f$ from $M$.
- $succ(\alpha, t_f) \neq \emptyset$ then $succ(\alpha, t_f) = (M', F')$, where: $\forall p \in P, M'(p) = M(p) - pre(p, t_f) + post(p, t_f)$ and $F'$ is computed in 3 steps:
  1) Set $F'$ to $F \land \bigwedge_{t \in En(M)} L_f \leq L \land \bigwedge_{t \in Nw(M, t_f)} \downarrow Is(t) \leq L - L_j \leq \uparrow Is(t)$. Notice that without loss of generality, for economy of notations, we suppose that the transitions of $En(M)$ are different from those newly enabled by transition $t_f$ from $M$.
  2) Put $F'$ in canonical form.$^2$
  3) Eliminate $L_f$ and all variables associated with transitions of $CF(M, t_f) - \{t_f\}$ and rename $L_j$ in $L_f$.

Canonical forms make operations over DBMs much simpler [11]. Two state classes are said to be equal iff they have the same canonical form (i.e., they have the same marking and the DBMs of their formulas have the same canonical form). Note that, in the following, we will use indifferently $(M, F)$ or $(M, D)$ to refer to the state class $\alpha$, and we suppose that all DBMs are in canonical form. DBM canonical forms allow also to reduce the complexity of the firing rule as follows [10].

Let $\alpha = (M, D)$ be a state class and $t_f \in T$ a transition.

- $t_f$ is fireable from $\alpha$ iff $t_f \in En(M) \land \forall t_i \in En(M), d_{ij} \geq 0$.
- If $t_f$ is fireable from $\alpha$ then its successor state class by $t_f$ is the state class $\alpha' = (M', D')$, where $M'$ and the canonical form of the DBM of $D'$ are computed as follows:
  \[
  \forall p \in P, M'(p) = M(p) - pre(p, t_f) + post(p, t_f) \land \forall t_i, t_j \in En(M'),
  \]
  \[
  d'_{i0} = \begin{cases} \uparrow Is(t_i) & \text{if } t_i \in Nw(M, t_f), \\ d_i & \text{if } t_i \notin Nw(M, t_f), \end{cases}
  \]
  \[
  d'_{ij} = \begin{cases} \downarrow Is(t_j) & \text{if } t_j \in Nw(M, t_f), \\ \min_{t_i \in En(M)} d_{ij} & \text{if } t_j \notin Nw(M, t_f), \\ 0 & \text{if } i = j, \end{cases}
  \]

III. COST TIME PETRI NETS

A. Definition and semantics

A cost time Petri net (cTPN for short) is a time Petri net where a rate cost is associated with each place, giving the sojourn cost per time unit of each token in that place. Formally, a Cost Time Petri Net is a tuple $N_c = (P, T, pre, post, M_0, I_s, r)$ where:

- $N = (P, T, pre, post, M_0, I_s)$ is a TPN,
- $r : P \rightarrow \mathbb{Z}$ is a rate cost function that associates a rate cost with each place of the TPN.

Note that no cost is associated with the discrete firings of transitions; however, these costs can be added without affecting the results provided in this paper.

Let $N_c$ be a cTPN, $t \in T$ a transition and $M$ a marking of the cTPN. We denote by $rm(M)$ the rate cost of $M$:

\[
rm(M) = \sum_{p \in P} M(p) \times r(p).
\]

The rate costs of places can be defined as in [7] by a linear function over markings. We define the incidence rate cost of $t$ by:

\[
rt(t) = \sum_{p \in P} (post(p, t) - pre(p, t)) \times r(p).
\]

Intuitively, it represents the impact of firing $t$ on the rate cost of a marking.

The semantics of a cTPN $N_c = (P, T, pre, post, M_0, I_s, r)$ is the semantics of the TPN $N = (P, T, pre, post, M_0, I_s)$. However, the rate costs associated with places allow to compute different costs such as the costs of runs and the optimal costs of firing a sequence from a state or a state class of $N_c$.

B. Cost of a run

Let $s_1 = (M_1, I_1)$ be a state of $N_c$ and $\sigma = (M_1, I_1) \xrightarrow{\theta_1 I_1} (M_2, I_2) \cdots (M_n, I_n) \xrightarrow{\theta_n I_n} (M_{n+1}, I_{n+1})$ a run of $s_1$. The cost of $\sigma$ is defined by:

\[
\text{Cost}(\sigma) = \sum_{t=1}^{n} (\theta_t \times rm(M_t)).
\]
Let \( \tau_0 \) the date at which the state \( s_1 \) is reached and \( \tau_j = \tau_0 + \sum_{i=1}^{j} \theta_i \) be the firing date of the transition \( t_j \) in \( \sigma \), for \( j = 1, n \). Proposition 1 rewrites \( \text{Cost}(\sigma) \) by means of the firing dates and incidence rate costs of transitions of \( \sigma \) (see Fig. 1). As we will show, this form is more useful to deal with the optimal-cost problem in some cases. The optimal-cost of \( \text{Cost}(\sigma) \) can be also rewritten by means of the firing dates and the rate costs of markings as shown in Proposition 2.

**Proposition 1:**

\[
\text{Cost}(\sigma) = \text{rm}(M_1) \times (\tau_n - \tau_0) + \sum_{j=1}^{n-1} \text{rt}(t_j) \times (\tau_n - \tau_j).
\]

**Proof:** By definition, \( \text{Cost}(\sigma) = \sum_{i=1}^{n} (\theta_i \times \text{rm}(M_i)) \). For \( i = 2, n \), the rate cost of the successor marking \( M_i \) of \( M_{i-1} \) by \( t_i \) is \( \text{rt}(M_i) = \text{rm}(M_{i-1}) + \text{rt}(t_{i-1}) \). Therefore, for \( i = 2, n \),

\[
\text{rt}(M_i) = \text{rt}(M_1) + \sum_{j=1}^{i-1} \text{rt}(t_j).
\]

Then:

\[
\text{Cost}(\sigma) = \theta_1 \times \text{rm}(M_1) + \sum_{i=2}^{n} (\theta_i \times (\text{rm}(M_1) + \sum_{j=1}^{i-1} \text{rt}(t_j))).
\]

It can be developed and rewritten as follows:

\[
\theta_1 \times \text{rm}(M_1) + 
\theta_2 \times (\text{rm}(M_1) + \text{rt}(t_1)) + \ldots + 
\theta_n \times (\text{rm}(M_1) + \text{rt}(t_1) + \text{rt}(t_2) + \ldots + \text{rt}(t_{n-1})).
\]

Finally, \( \text{Cost}(\sigma) \) can be rewritten so as the rate cost of \( M_1 \) (i.e., \( \text{rm}(M_1) \)) is the coefficient of \( \sum_{i=1}^{n} \theta_i \), the incidence rate cost of \( t_1 \) (i.e., \( \text{rt}(t_1) \)) is the coefficient of \( \sum_{i=2}^{n} \theta_i \), and so on, the incidence rate cost of \( t_{n-1} \) (i.e., \( \text{rt}(t_{n-1}) \)) is the coefficient of \( \theta_n \). It follows that:

\[
\text{Cost}(\sigma) = \text{rm}(M_1) \times (\sum_{i=1}^{n} \theta_i) + \sum_{j=1}^{n-1} \text{rt}(t_j) \times (\sum_{i=j+1}^{n} \theta_i).
\]

To achieve the proof, it suffices to replace \( \sum_{i=1}^{n} \theta_i \) with \( \tau_n - \tau_0 \) and \( \sum_{i=j+1}^{n} \theta_i \) with \( \tau_n - \tau_j \).

**Proposition 2:**

\[
\text{Cost}(\sigma) = (\sum_{i=1}^{n-1} -\text{rt}(t_i) \times \tau_i) + \text{rm}(M_n) \times \tau_n.
\]

**Proof:** By definition, \( \text{Cost}(\sigma) = \sum_{i=1}^{n} (\theta_i \times \text{rm}(M_i)) \). For \( i = 1, n \), \( \theta_i = \tau_i - \tau_{i-1} \). Then:

\[
\text{Cost}(\sigma) = \sum_{i=1}^{n} ((\tau_i - \tau_{i-1}) \times \text{rm}(M_i)).
\]

For \( i = 1, n - 1 \), the rate cost of the successor marking \( M_{i+1} \) of \( M_i \) by \( t_i \) is \( \text{rm}(M_{i+1}) = \text{rm}(M_i) + \text{rt}(t_i) \). Therefore, for \( i = 1, n - 1 \), \( \text{rt}(t_i) = \text{rm}(M_{i+1}) - \text{rm}(M_i) \). The previous expression of \( \text{Cost}(\sigma) \) can be developed and rewritten as follows:

\[
(\sum_{i=1}^{n-1} (\text{rm}(M_i) - \text{rm}(M_{i+1}) \times \tau_i) + \text{rm}(M_n) \times \tau_n.
\]

To achieve the proof, it suffices to replace \( \text{rm}(M_i) - \text{rm}(M_{i+1}) \) with \( -\text{rt}(t_i) \).

**C. Optimal cost of a firing sequence**

Let \( \alpha_1 \) be a state class of \( \mathcal{N}_c \) and \( \rho = \alpha_1 \xrightarrow{t_1} \alpha_2 \cdots \alpha_n \xrightarrow{t_n} \alpha_{n+1} \) a path of \( \alpha_1 \), \( \omega = t_1 \cdots t_n \) and \( \Pi(\alpha_1, \omega) \) the set of runs of \( \alpha_1 \) that support the same sequence of transitions \( \omega \) and lead to states of \( \alpha = \alpha_{n+1} \). The optimal-cost of firing \( \omega \) from \( \alpha_1 \) (or the optimal-cost of \( \rho \)) is:

\[
\text{OptCost}(\alpha_1, \omega) = \min_{\sigma \in \Pi(\alpha_1, \omega)} \text{Cost}(\sigma).
\]

The optimal-cost of firing \( \omega \) from \( \alpha_1 \) can be computed by extending state classes with costs and using linear programming techniques as in [7].

**D. Optimal-cost reachability problem**

The classical forward exploration algorithm in [2]–[4], [7] is adapted in Algorithm 1 to compute the optimal-cost to reach, from the initial marking, a marking belonging to a given set of markings \( \text{Goal} \). For each state class \( \alpha \) such that its marking is in \( \text{Goal} \), its optimal-cost is computed and compared with \( \text{MinCost} \), where the smallest cost computed so far is saved. As usual, the lists \( \text{Passed} \) and \( \text{Waiting} \) are used to store the already processed and not yet processed state classes, respectively. The notation \( \omega < \omega' \) means that \( \omega \) is a prefix of \( \omega' \).

For bounded TPNs with no negative cost cycles, the algorithm terminates as for infinite sequences only finite prefixes, yielding the longest elementary paths, are explored.

In the following sections, we investigate cases where the optimal-costs of firing sequences can be computed without splitting state classes nor using linear programming techniques.

**IV. Computing optimal cost of firing sequences**

Let \( \alpha_1 = (M_1, F_1) \) be a state class of \( \mathcal{N}_c \) and \( \rho = \alpha_1 \xrightarrow{t_1} \alpha_2 \cdots \alpha_n \xrightarrow{t_n} \alpha_{n+1} \) a path of \( \alpha_1 \) and \( \omega = t_1 \cdots t_n \). For ease of notation, we suppose that all transitions within \( E(M_1) \) and those enabled by every transition of \( \omega \) are all different. The firing date domain of transitions of \( \omega \) from \( \alpha_1 \) can be retrieved by modifying the firing rule given in Section II-B. Indeed, it suffices, in step 3 of \( \text{succ}(\alpha_1, t_i) \), for \( i \in [1, n] \), to keep \( t_i \). More precisely replace step 3 with: Eliminate all variables associated with transitions of \( \text{CF}(M_i, t_i) - \{t_i\} \). With this modification, each variable \( t_i \) for \( i \in [1, n] \), represents the firing date of the \( i^{th} \) transition of \( \omega \). The variable \( t_0 \) is the
Algorithm 1 Symbolic algorithm for optimal-cost reachability problem

1: \( \text{MinCost} \leftarrow \infty \)
2: \( \text{Passed} \leftarrow \emptyset \)
3: \( \text{Waiting} \leftarrow \{((M_0, D_0), \epsilon)\} \)
4: while \( \text{Waiting} \neq \emptyset \) do
5: \( (M, D, \omega) \leftarrow \text{Waiting} \)
6: if \( M \in \text{Goal} \) and \( \text{OptCost}((M_0, D_0), \omega) < \text{MinCost} \) then
7: \( \text{MinCost} \leftarrow \text{OptCost}((M_0, D_0), \omega) \)
8: end if
9: if (for all \( ((M', D'), \omega') \in \text{Passed} \), \( (M', D') \neq (M, D) \) or \( \omega' < \omega \)) then
10: add \( ((M, D), \omega) \) to \( \text{Passed} \)
11: for all \( t \in \text{Fr}(M, D) \), add \( \text{succ}((M, D), t, \omega t) \) to \( \text{Waiting} \)
12: end if
13: end while
14: return \( \text{MinCost} \)

\[ \text{date when } \alpha_1 \text{ is reached. Let } FG \text{ be the resulting formula.} \]

The firing date domain of transitions of \( \omega \) from \( \alpha_1 \) is the projection of the domain of \( FG \) to \( \mathbb{L}_i \) for \( i \in [0, n] \).

A. Case of non-negative incidence rate costs

For this section, we suppose that the incidence rate costs of all transitions of \( \omega \) except the last one are non-negative and \( \text{rm}(M_1) \geq 0 \). We will show that, under these assumptions, to compute the optimal-cost of firing \( \omega \) from \( \alpha_1 \), we need to keep track of the minimal delay between the previously fired transitions and the coming ones, including the current one. But we do not need to retrieve delays between the previously fired transitions. Consequently, the relevant part of \( FG \) needed to compute the optimal-costs can be represented by a DBM of order \( (\omega + 1) \times |\text{En}(M) \cup \{t_n\}| \).

We denote by \( G \) the DBM in canonical form of order \( (\omega + 1) \times |\text{En}(M) \cup \{\mathbb{L}_n\}| \) defined by \( \forall i \in [0, |\omega|], \forall t_j \in \text{En}(M) \cup \{t_n\}, g_{ij} = \text{Max}(\mathbb{L}_i - \mathbb{L}_j|FG) \). Since \( \mathbb{L}_i - \mathbb{L}_j \leq 0 \), the value \( -g_{ij} \) is the minimal delay between the firing dates \( t_j \) and \( t_i \). Note that \( G \) is a sub-matrix of the DBM in canonical form of \( FG \). The size of the DBM of \( FG \) is \( (\omega + 1) \times |\text{En}(M)|^2 \).

**Theorem 1:**

\[ \text{OptCost}(\alpha_1, \omega) = -g_{0n} \times \text{rm}(M_1) + \sum_{i=1}^{n-1} \text{rt}(t_i) \times -g_{in}. \]

**Proof:** Let \( GG \) be the DBM in canonical form of \( FG \). Recall that \( G \) is a sub-matrix of \( GG \). To achieve the proof, we first show that the valuation \( v_i = g_{in} - g_{0n} \) for \( i \in [1, n] \) is a feasible firing schedule for \( \omega \), i.e., for \( i, k \in [1, n] \), \(-g_{0n} \leq v_i \leq g_{0n} \) and \( v_i - v_k \leq g_{ik} \).

By definition, for \( i \in [1, n] \), \( v_i = g_{in} - g_{0n} = g_{in} - g_{0n} \), then for \( i, k \in [1, n] \), \( v_i - v_k = g_{in} - g_{kn} = g_{in} - g_{0n} \). Since \( GG \) is in canonical form, it holds that \( g_{in} \leq g_{in} + g_{0n} \), \( g_{in} \leq g_{in} + g_{0n} \), and \( g_{in} \leq g_{in} + g_{0n} \). It follows that \(-g_{0n} \leq v_i \leq g_{0n} \) and \( v_i - v_k \leq g_{ik} \).

The run corresponding to this firing schedule of \( \omega \) is shown in Fig. 2. Its cost is:

\[ -g_{0n} \times \text{rm}(M_1) + \sum_{i=1}^{n-1} \text{rt}(t_i) \times -g_{in}. \]

According to the definition of \( G \), for \( \omega = \epsilon \), \( G \) is a DBM of order \( 1 \times (|\text{En}(M)| + 1) \) defined by \( \forall t_j \in \text{En}(M) \cup \{t_0\}, g_{0j} = d_{0j} \). Let us show now how to compute progressively the DBM \( G \) of a nonempty sequence.

**Proposition 3:** Let \( \alpha \) be the state class reached by a path \( \rho \), the corresponding DBM and \( t_f \) a transition firable from \( \alpha \) and \( \alpha' = (M', D') = \text{succ}(\alpha, t_f) \).

Then, the DBM \( G' \) of the path \( \rho \rightarrow (M', D') \) is the DBM in canonical form of order \( (|\omega t_f| + 1) \times |\text{En}(M') \cup \{t_f\}| \) defined by:

\[ \begin{align*}
M_{in}(g_{ij}, g_{if} + d_{0j}) & \quad \text{if } i \leq |\omega| \wedge t_j \notin \text{Nw}(M, t_f) \\
g_{if} + d_{0j} & \quad \text{if } i \leq |\omega| \wedge t_j \in \text{Nw}(M, t_f) \\
d_{0j} & \quad \text{if } i = |\omega t_f| 
\end{align*} \]

where \( d_{0j} = \begin{cases} M_{in}(g_{ij}, g_{if} + d_{0j}) & \text{if } t_j \notin \text{Nw}(M, t_f) \\
M_{in}(d_{0j}) & \text{if } t_j \in \text{Nw}(M, t_f) \end{cases} \) and \( d_{0j} = -\downarrow \text{Is}(t_f) \), otherwise.

**Proof:** The proof is based on the constraints added to compute \( \text{succ}(\alpha, t_f) \). Indeed, the firing condition is obtained by adding to \( D \), the constraints:

\[ \downarrow \text{Is}(t_f) \leq \mathbb{L}_f, \text{ for } t_n \in \text{En}(M) \] and \( \downarrow \text{Is}(t_f) \leq \mathbb{L}_f \leq \downarrow \text{Is}(t_f), \text{ for } t_j \in \text{Nw}(M, t_f) \).
Notice that all the added constraints involve \( t_f \) and in the corresponding constraint graph, they are represented by arcs \( \langle \mathcal{L}_f, \mathcal{L}_f, 0 \rangle \), for \( t_u \in En(M) \), \( \langle \mathcal{L}_j, \mathcal{L}_f, \downarrow Is(t_j) \rangle \) and \( \langle \mathcal{L}_f, \mathcal{L}_j, \neg \downarrow Is(t_j) \rangle \), for \( t_j \in Nw(M, t_f) \).

Therefore, the shortest path connecting a node \( \mathcal{L}_i \), for \( i \in [0, |\omega t_f|] \), to a node \( \mathcal{L}_j \in En(M') \) is:

\[
\min(\sum_{t_w \in En(M)} d_{\omega_{ij}} + \sum_{t_u \in Nw(M, t_f)} g_{ij} - \downarrow Is(t_j), \text{if } t_j \notin Nw(M, t_f) \),
\]

By the firing rule given in II-B, it holds that: \( d_{ij} = \left\{ \begin{array}{ll}
\downarrow Is(t_j) & \text{if } t_j \in Nw(M, t_f), \\
\min(\sum_{t_u \in En(M)} d_{\omega_{ij}} + \sum_{t_u \in En(M)} g_{ij} - \downarrow Is(t_j), \text{if } t_j \notin Nw(M, t_f) \).
\end{array} \right. \]

Note that, \( \min(\sum_{t_u \in Nw(M)} d_{\omega_{ij}}, \sum_{t_u \in Nw(M)} g_{ij} - \downarrow Is(t_j)) = g_{ij} \), for \( i \in [0, |\omega t_f|] \).

For \( i = |\omega t_f| \), \( g_{ij}' = d_{ij} \) as \( g_{ij}' \) is the smallest path connecting \( \mathcal{L}_f \) to \( \mathcal{L}_j \) \( \neg \downarrow Is(t_j) \).

The computation of the optimal-cost of firing \( \omega \) from \( \alpha_0 \) needs to carry in the DBM \( G \) the minimal firing delay between each fired transition of \( \omega \) and the coming ones, including the current one. Thus, the size of \( G \) grows with the size of \( \omega \) indeed, the optimal-cost of \( |\omega t_f| \) is reached when \( t_f \) is fired as soon as possible (i.e., \( g_{0f} = -d_{0f} \)) from \( \alpha_0 \) and the previous ones are fired as late as possible (i.e., \( g_{ij} \)) without causing any delay to \( t_f \). It means that, to retrieve the firing schedule yielding the optimal-cost of \( \omega \), the firing dates of its previous transitions need to be updated to take into account the fact that \( t_f \) is fired as soon as possible and the previous ones are fired as late as possible before \( t_f \).

However, for bounded TPNs with no negative cost cycles, each infinite sequence \( \omega' \) of \( N_c \), has some prefix \( \omega \) followed by a repetitive sequence \( \omega' \) that loops on a state class \( \alpha \) reachable from the initial state class \( \alpha_0 \) by \( \omega \). It follows that \( OptCost(\alpha_0, \omega) \leq OptCost(\alpha_0, \omega') \) and \( OptCost(\alpha, \omega) \leq OptCost(\alpha, \omega') \) for \( k \geq 1 \). The cost of a cycle is non-negative, if the sum of the incidence rate costs of all its transitions is non-negative and the rate cost of one of its marking is non-negative. This sufficient condition guarantees for a cycle a non-negative cost.

In the following, we investigate the possibility to reduce the size of the DBM \( G \).

B. Memoryless state classes w.r.t. optimal-costs

Let \( \alpha = (M, D) \) be the state class reached by a sequence \( \omega \) from a state class \( \alpha_1 \) of \( N \). The state class \( \alpha \) is said to be memoryless w.r.t. optimal-costs iff for each sequence \( \omega' \) of \( \alpha \), \( OptCost(\alpha_1, \omega') = OptCost(\alpha_1, \omega) + OptCost(\alpha, \omega') \).

Therefore, for any state classes \( \alpha_1 \) and \( \alpha_1' \) leading by sequences \( \omega \) and \( \omega' \), respectively, to the same memoryless state class \( \alpha = (M, D) \) w.r.t. optimal-costs, it holds that for each fireable sequence \( \omega'' \) from \( \alpha_1 \),

\[
OptCost(\alpha_1, \omega) \leq OptCost(\alpha_1', \omega') \Rightarrow OptCost(\alpha_1, \omega'') \leq OptCost(\alpha_1', \omega'')
\]

Lemmas 1 and 2 provide two different sufficient conditions for the state class \( \alpha \) to be memoryless w.r.t. optimal-costs. The first one depends on the DBM \( G \) of \( \omega \). The second one depends on \( \alpha \).

\textbf{Lemma 1:} Let \( \alpha = (M, D) \) be the state class reached by some sequence \( \omega = t_1 \cdots t_n \) from a state class \( \alpha_1 = (M_1, D_1) \) and \( G \) the DBM of its firing domain.

If \( \forall i \in [0, |\omega|], \exists t_j \in En(M), g_{ij} = g_{in} + d_{ij} \) then, \( \alpha \) is memoryless w.r.t. optimal-costs.

\textbf{Proof:} Suppose that \( \forall i \in [0, |\omega|], \forall t_j \in En(M), g_{ij} = g_{in} + d_{ij} \). Let us show that for every sequence \( \omega' \) of \( \alpha \), \( OptCost(\alpha_1, \omega') = OptCost(\alpha_1, \omega) + OptCost(\alpha, \omega') \). We consider 2 cases: 1) \( \omega = t_1 \cdots t_n \) and \( \omega' = t_j \), and 2) \( \omega = t_1 \cdots t_n \) and \( |\omega'| > 1 \).

1) Case \( \omega = t_1 \cdots t_n \) and \( \omega' = t_j \): Since \( \forall i \in [0, |\omega|], g_{ij}' = g_{in} + c_{ij} \), Consequently, if \( t_j \) is firable from \( \alpha = (M, D) \) then the optimal-cost of the successor of \( \alpha \) by \( t_j \) is:

\[
OptCost(\alpha_1, \omega) + (rm(M_1) + \sum_{i \in [1, |\omega|]} rt(\omega(t_i))) \times -d_{ij}.
\]

Note that \( rm(M) = rm(M_1) + \sum_{i \in [1, |\omega|]} rt(\omega(t_i)) \) and \( rm(M) \times -d_{ij} \) is the optimal-cost of firing \( t_j \) from \( (M, D) \).

2) Case \( \omega = t_1 \cdots t_n \) and \( |\omega'| > 1 \): Suppose that in the DBM \( G' \) of \( \omega' \) from \( \alpha_0 \), \( \forall t_j \in En(M'), g_{ij}' = g_{in} + c_{ij} \), where \( c_{ij} \) does not depend on \( i \) and is the opposite of the minimal delay between firing dates of \( t_j \) and \( t_n \). Let us show that in any extended sequence of \( \omega' \) by any firable transition \( t_k \) leading to the state class \( (M''', D''') \), it holds that \( OptCost(\alpha_1', \omega') \leq OptCost(\alpha_1, \omega'') \) for \( k \geq 1 \). The cost of a cycle is non-negative, if the sum of the incidence rate costs of all its transitions is non-negative and the rate cost of one of its marking is non-negative. This sufficient condition guarantees for a cycle a non-negative cost.

In the following, we investigate the possibility to reduce the size of the DBM \( G \).
OptCost

Therefore, \( \omega \) is:

By Proposition 2, the cost of each run supporting the sequence \( \alpha \) domain. As all transitions of \( \alpha \) are newly enabled, it follows that \( \forall i \in [0, |\omega|], \forall t_j \in E_n(M), g_{t_j} = g_{t_i} + d_{t_0} \). According to Lemma 1, \( \omega \) is memoryless w.r.t. optimal-costs.

Thanks to lemmas 1 and 2, when a memoryless state class \( \alpha \) w.r.t. optimal-costs is reached, there is no need to explore its successors, in case there is in the list \( \mathcal{P} \) an identical memoryless state class with smaller optimal-cost. Among the identical memoryless state classes reached by different paths, the one with the smallest optimal-cost will yield the optimal reachable cost, for all state classes reachable from \( \alpha \). Therefore, Algorithm 1 can be improved for this subclass of cTPNs.

C. Case of non-positive incidence rate costs

For this section, we suppose that the incidence rate costs of all transitions of \( \omega \), except the last one, are non-positive and \( rm(M_n) \geq 0 \).

Theorem 2: Let \( \rho = \alpha_1 = (M_1, D_1) \xrightarrow{t_1} \alpha_2 = (M_2, D_2) \cdots \alpha_n = (M_n, D_n) \xrightarrow{t_n} \alpha_{n+1} = (M_{n+1}, D_{n+1}) \) be a path in the SCG of \( \mathcal{N}_c \), supporting the sequence \( \omega = t_1 \cdots t_n \). Let \( GG \) be the DBM in canonical form of \( FG \) (the firing domain transitions of \( \omega \) and those enabled in \( M_n \)). Then,

\[
OptCost(\alpha_1, \omega) = \left( \sum_{i=1}^{n} rt(t_i) \times g_{g(t_i)} \right) - g_{g(t_n)} \times rm(M_n).
\]

Proof: We first show that the valuation \( v_i = -g_{g(t_i)} \) for \( i \in [0, n] \) is a feasible firing schedule for \( \omega \), i.e., for \( i, k \in [1, n] \), \( -g_{g(t_i)} \leq v_i \leq g_{g(t_i)} \) and \( v_i - v_k \leq g_{g(t_k)} \).

By definition, for \( i \in [1, n] \), \( v_i = -g_{g(t_i)} \), then for \( i, k \in [1, n] \), \( v_i - v_k = g_{g(t_i)} - g_{g(t_k)} \). Since \( GG \) is in canonical form, it holds that \( g_{g(t_i)} \leq g_{g(t_k)} + g_{g(t_k)} \). It follows that \( -g_{g(t_i)} \leq v_i \leq g_{g(t_i)} \) and \( v_i - v_k \leq g_{g(t_k)} \).

By assumption, \( rm(M_n) \geq 0 \) and for \( i \in [1, n] \), \( rt(t_i) \leq 0 \). Furthermore, for \( i \in [0, n] \), \( -g_{g(t_i)} \) is the minimal value of \( \tau_i \) in \( FG \) (i.e., the firing date of the \( i^{th} \) transition of \( \omega \)).

By Proposition 2, the cost of each run supporting the sequence \( \omega \) is:

\[
\left( \sum_{i=1}^{n-1} -rt(t_i) \times \tau_i \right) + rm(M_n) \times \tau_n.
\]

Therefore, \( OptCost(\alpha_1, \omega) = \left( \sum_{i=1}^{n-1} rt(t_i) \times g_{g(t_i)} \right) - rm(M_n) \times g_{g(t_n)} \).

D. Case of singular intervals

For this section, we suppose that the firing intervals of all transitions of \( \omega \) are singular but the incidence cost rate of each transition of \( \omega \) is negative, null or positive.

Theorem 3: Let \( \rho = \alpha_1 = (M_1, D_1) \xrightarrow{t_1} \alpha_2 = (M_2, D_2) \cdots \alpha_n = (M_n, D_n) \xrightarrow{t_n} \alpha_n = (M_{n+1}, D_{n+1}) \) be a path in the SCG of \( \mathcal{N}_c \), supporting the sequence \( \omega = t_1 \cdots t_n \). Then,

\[
OptCost(\alpha_1, \omega) = \sum_{i=1}^{n} (-d_{i_{0i}} \times rm(M_i)).
\]

Proof: The cost of each run \( \sigma = (M_1, I_1) \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} (M_{n+1}, I_{n+1}) \) supporting \( \omega \) is:

\[
\sum_{i=1}^{n} \left( \theta_i \times rm(M_i) \right).
\]

The domain of each \( \theta_i \), for \( i \in [1, n] \), \([-d_{i_{0i}}, d_{i_{0i}}]\). As the firing interval of all transitions of \( \omega \) are singular, each transition is fired at an exact date. Therefore, for \( i \in [1, n] \), \(-d_{i_{0i}} = d_{i_{0i}}\).

V. CASE STUDY

In the French academic system, faculty positions with both teaching and research activities can be held either by an associate professor (maître de conférences, or MCF) or by a full professor (professeur des universités, or PU). In this system, a typical career path is:

- start as an associate professor at the 1st grade (échelon 1);
- get promoted to higher grades; such promotions are automatic and usually happen every 34 months (that is, 2 years and 10 months after the last promotion);
- after some years, defend a habilitation thesis and obtain a higher degree (known as the habilitation à diriger des recherches), a qualification needed to supervise PhD students;
- depending on the opportunities, get a promotion from associate to full professor; keep getting automatic promotions to higher grades according to seniority.

To each grade corresponds an indice (a salary scale grade), on which the salary is based; additionally, when promoted from associate to full professor, the indice cannot decrease.

Let us consider the case of an associate professor who reaches the 4th échelon when 32 years old. Let us further

\(^3\)For some reason, the switch from 1st to 2nd grade only takes 12 months, whereas the switch from 6th to 7th takes 42 months (3 years and 6 months).

\(^4\)After a while, échelons are called chevrons or stripes.
suppose that the university wishes that all faculty members become full professors and reach the last grade by the time they are 55. Obviously, the cheapest way for the university to reach this goal is to promote anyone at the latest possible time: given the durations between grades, this means letting the person reach the 9th grade after 14 years and 10 months; keep them in that grade for 2 years and 8 months; promote them to full professor and let them reach the last grade after 5 years and 6 months.

However, this strategy does not take into account the frustration of the person, which increases each time a promotion is denied, starting from the moment they reach the 6th grade and begin questioning their life choices.

We propose a model for this optimisation problem, shown in Fig.5. Each place with a $MCF_{xyz}$ label corresponds to a grade in the associate professor scale; its rate cost is equal to $xyz$ (and is actually equal to the indice for this grade). Similarly, each place with a $PU_{xyz}$ label corresponds to a grade in the full professor scale. In the following, we give various values to $R = r(unhappy)$ so as to show the interest of our method; the rate cost of all the other places is set to 0.

The state class graph of the model is depicted in Fig.4 and Table I; note that a total of 10 paths in the SCG lead to a goal containing 1 token; in the following, we denote Goal the set of such markings.

To keep the model simple enough, it should be noted that it does not guarantee that the place goal is attained when the person is exactly 55 years old (a token could stay in place wait for more than 0 time unit, for instance). However, we can show that, in the following, all optimal-cost strategies are such that the place goal is attained as early as possible, that is, after 23 years.

Let us first set $R = 0$. The optimal cost of each path leading to a marking in Goal is given in Table II and the minimum is equal to 208.668. The computation steps of this cost are reported in Fig.5: as expected, the optimal strategy is to give the promotion at the latest possible time, thus leading to the following timed trace: $ecelon5@34, up2@34, PUech3@12, PUech4@12, PUech5@12, PUech6@42, chevron2@12, chevron3@12, age55years@0, end@0$.

Let us now change the value of $R$; whenever $R \leq 32$, the optimum strategy remains the same. For $R = 33$, the strategy consists in giving the promotion not too early, just before switching from 6th to 7th grade. The timed trace, with a total cost of 228.480, is: $ecelon5@34, up2@34, PUech3@12, PUech5@12, PUech6@42, chevron2@12, chevron3@12, age55years@106, end@0$.

Whenever $R \geq 34$, the strategy consists in giving the promotion just before risking unhappiness, that is right before switching from 5th to 6th grade. For $R = 35$, the computation steps are reported in Fig.6 and the timed trace, with a total cost of 228.660, is: $ecelon5@34, up2@34, PUech3@12, PUech4@12, PUech5@12, PUech6@42, chevron2@12, chevron3@12, age55years@106, end@0$.

VI. Conclusion

This paper deals with the optimal-cost reachability problem in the context of time Petri Nets extended with costs (cTPNs). It establishes, for some interesting subclasses of cTPNs, efficient algorithms that compute the optimal-cost of firing a sequence of transitions from a given state class. Unlike the approaches developed in [1]–[7], the algorithms, presented here, are not based on techniques of linear programming. Finally, a case study is provided to show the interest of the proposed method.

As a future work, we will investigate the optimal-cost reachability problem in the context of parametric cTPNs.

References


Fig. 3. Possible career paths from age 32 to 55

Fig. 4. The SCG of the TPN at Fig.3

Fig. 5. Optimal-cost of the sequence (echelon5 · · · echelon9 up6 PUech6 chevron2 chevron3 age55years end) is 20 8 668
TABLE I
STATE CLASSES OF THE SCG IN FIG.4

<table>
<thead>
<tr>
<th>Path</th>
<th>Optimal-Cost</th>
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</thead>
<tbody>
<tr>
<td>$P_{Uech}$</td>
<td>$228$</td>
</tr>
<tr>
<td>$P_{Tech}$</td>
<td>$219$</td>
</tr>
<tr>
<td>$P_{Uech}$</td>
<td>$208$</td>
</tr>
<tr>
<td>$P_{Tech}$</td>
<td>$131$</td>
</tr>
<tr>
<td>$P_{Uech}$</td>
<td>$104$</td>
</tr>
<tr>
<td>$P_{Uech}$</td>
<td>$85$</td>
</tr>
</tbody>
</table>

TABLE II
OPTIMAL-COST OF EACH PATH OF THE SCG IN FIG.4 THAT LEADS TO A MARKING IN Goal