A FEM-based Nonlinear MAP Estimator in Electrical Impedance Tomography

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Abstract

Electrical Impedance Tomography *of closed conductive media is an ill-posed inverse problem. Using the* Finite Elements Method *to solve the corresponding direct problem allows to preserve the nonlinear dependence of the observation set upon the conductivity distribution. In this paper, we show that the Bayesian approach presented in* [1] *for linear inverse imaging problems is still valid for such a non linear inverse problem. Our contribution is based on an edge-preserving Markov model as prior for conductivity distribution.* Maximum A Posteriori *reconstruction results from 40-dB noisy measurements (simulated with a finer mesh) yield significant resolution improvement compared to classical methods.*

1 Introduction

Electrical Impedance Tomography (EIT) of closed conductive media with steady currents is a non-invasive imaging technique that aims at estimating the impedance distribution within a conductive body from electrical measurements on the surface. Applications can be found in medical imaging and non-destructive testing. Usually, thanks to surface electrodes, the experiment consists in injecting currents in the body and measuring the surface voltage distributions. Here, we deal with 2D reconstructions.

Contrary to many inverse imaging problems such as deconvolution or Fourier synthesis [1], EIT stems from nonlinear observation equations. Indeed, the current paths throughout the medium (and thus the observations) highly depend on the conductivity distribution. The image-data relation, derived from Maxwell's formulas, is ruled by a second order partial derivative equation. However, since this equation yields no analytical solution for arbitrary conductivity distribution, one has to approximate the direct problem. Many authors [2, 3] proposed linear approximations. However, if the sought distribution is highly contrasted (which is often the case in EIT applications), linearized models are no longer valid.

Following [4], we use a *Finite Element Method* (FEM) direct model, which preserves the non linear dependence of the observations upon the conductivity distribution. FEM convergence properties towards the true solution are well known, and this technique is frequently used in many problems involving partial differential equations (PDE), in such fields as electrostatics, solid or fluid mechanics... Besides, discretization of the 2D domain into elements reveals as also suited to a Bayesian approach of inversion, as shown in section III.

One of the major difficulties of EIT is its ill-posed character [5], which was ignored by the first reconstruction methods [2], based on simple backprojection techniques. As well as other methods developed so far, they provide unregularized estimators that are very sensitive to noise (even with a SNR below 70 dB). In [6], we showed that the Bayesian approach presented in [1] was also a valid frame for ill-posed EIT. In order to reconstruct an image of *log-conductivity* elements within a circular domain (where each element enters the FEM triangle mesh system), we considered a neighborhood system compatible with the mesh system. This allowed the introduction of a Markov Random Field as a prior model for log-conductivity. With a 60 dB signal-to-noise ratio (SNR) and highly contrasted conductivity distribution, *Maximum A Posteriori* MAP reconstruction results, obtained through the optimization of the posterior likelihood criterion, yield significant resolution improvement compared to classical methods.

In this paper, we take advantage of the robustness of this method. First, in order to get more realistic observation sets, we use a finer domain discretization to generate the observed voltage outputs, keeping the coarser resolution to the MAP estimation method. We try here to estimate intermediate contrasted conductivity distribution, on a non convex domain, with a much lower SNR: 40 dB.

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2 FEM for the direct model

Let us call Ω the 2D domain and $\bar{\Omega}$ its border. We respectively note I, V the current stream and voltage distributions on Ω and $\bar{I}$, $\bar{V}$ the same distributions restricted on $\bar{\Omega}$. Lastly, we call σ the conductivity distribution. On one hand, the direct problem consists in finding $\bar{V}$ from $\bar{I}$ and σ, while the inverse problem requires the determination of σ from $\bar{I}$ and $\bar{V}$.

The observation model is described by the PDE $\mathrm{div}(\sigma\overrightarrow{\mathrm{grad}}V) = 0$, along with adequate additive current boundary conditions. In the general case, there exists no analytical solution for this PDE [7]. When linear approximations are too rough, the equivalent following integral formulation of the problem

$$\frac{1}{2}\iint_{\Omega}[\sigma|\overrightarrow{\mathrm{grad}}V|^2 + \sigma^{-1}|\overrightarrow{\mathrm{grad}}I|^2]\,dx\,dy + \int_{\bar{\Omega}} V\frac{\partial \bar{I}}{\partial \vec{t}}\,ds = 0$$

can be discretized with the Finite Element Method [8, 9, 4, 10]. The discretization of the problem amounts to divide the domain into, say P elements defining N nodes on Ω, among which $\bar{N}$ are on the border $\bar{\Omega}$. Thereafter, the discretized observation model reads:

$$\bar{\boldsymbol{v}} = \boldsymbol{P}\boldsymbol{A}_{\boldsymbol{\sigma}}^{-1}\boldsymbol{P}^\top \boldsymbol{D}\bar{\boldsymbol{\imath}}, \tag{1}$$

where $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{\imath}}$ (of size $\bar{N}$, *i.e.* the number of border nodes) are discretized counterparts of $\bar{V}$ and $\bar{I}$. The vector $\boldsymbol{\sigma}$ contains the P elements of conductivity. $\boldsymbol{A}_{\boldsymbol{\sigma}}$ is the $(N-1)\times(N-1)$ « stiffness » matrix of the problem defined by:

$$a_{ij} = \sum_{n\in e(i)\cap e(j)} \sigma_n\ \theta(i,j,n), \tag{2}$$

where $e(i)$ is the set of neighboring elements of node i and $\theta(i,j,p)$ depends on the geometric features of the mesh on Ω. The matrix $\boldsymbol{A}_{\boldsymbol{\sigma}}$ is positive-definite and very sparse: in practice, more than 98% of its entries are null. $\boldsymbol{P}$ is merely a $\bar{N}\times(N-1)$ border projection operator and $\boldsymbol{D}$ is a difference operator defined by:

$$(\boldsymbol{D}\bar{\boldsymbol{\imath}})_n = \bar{\imath}_{n+1} - \bar{\imath}_{n-1}\ [\bar{N}], \tag{3}$$

i.e., the current stream difference at the border, measured between the two neighbors of the n^{th} node. The matrix $\boldsymbol{A}_{\boldsymbol{\sigma}}$ linearly depends on $\boldsymbol{\sigma}$, so the FEM image-data relation Eq.(1) is a *non linear* function of the conductivity distribution. Actually, in our FEM discretization, current stream and voltages are approximated by piecewise linear functions on each element, whose nodal values are respectively the components of $\bar{\boldsymbol{\imath}}$ et $\bar{\boldsymbol{v}}$ while conductivity is approximated through piecewise constant functions whose element values belong to $\boldsymbol{\sigma}$.

3 *Maximum A Posteriori* estimator

The condition number of $\boldsymbol{A}_{\boldsymbol{\sigma}}$ does not exceed 10^4 when $P < 1000$, so Eq. (1) provides a numerically stable solution to the direct problem. The knowledge of $\bar{\boldsymbol{\imath}}$ ($\bar{N}\times 1$) and $\boldsymbol{\sigma}$ ($P\times 1$) allows to compute $\bar{\boldsymbol{v}}$ ($\bar{N}\times 1$). On the other hand, there exists no straight inverse way to get $\boldsymbol{\sigma}$ from the observation. First of all, since we have $\bar{N} < N < P$, the number of data $\bar{N}$ is lower than the number of unknows P. In practice, K independent observation sets are gathered, with $K > P/\bar{N}$, so that classical backprojection techniques are implementable [2, 10]. Such methods aim at minimizing the quadratic distance between the observations and the FEM direct simulation. Yet, these methods provide no acceptable results. This is not surprising since, we guess, a small change on inner conductivity has very few influence on the observation on the border. On the other hand, given the ill-posed character of the inverse problem, the backprojected image may rather depict amplified noise rather than true conductivity values.

Thus, in 1991, Hua *et al.* [4] proposed to add a quadratic penalty term to the usual criterion. Its stabilizing effect is well-known and provides robust conductivity maps, but at the expense of poor resolution. Here we propose to introduce another kind of penalty term, corresponding to a non-Gaussian Markov prior probability from the Bayesian point of view. On the other hand, following Barber and Brown [3], the use of the log-conductivity $\gamma = log(\sigma)$ is preferred. Unlike σ, it needs not be constrained positive. Besides, the mapping $\bar{\boldsymbol{v}}(\gamma)$ looks closer to linearity than $\bar{\boldsymbol{v}}(\boldsymbol{\sigma})$. Both features make the penalized criterion easier to optimize with respect to γ. It is defined as the Bayesian posterior likelihood:

$$p(\gamma|\bar{\boldsymbol{v}}^k;\bar{\boldsymbol{\imath}}^k) \propto p(\bar{\boldsymbol{v}}^k|\gamma;\bar{\boldsymbol{\imath}}^k)p(\gamma). \tag{4}$$

If we consider a centered white Gaussian noise $\bar{\boldsymbol{n}}^k$ of variance λ^2, for the K direct observation models $k = 1,\ldots,K$:

$$\bar{\boldsymbol{v}}^k = \boldsymbol{P}\boldsymbol{A}_{\boldsymbol{\sigma}}^{-1}\boldsymbol{P}^\top\boldsymbol{D}\bar{\boldsymbol{\imath}}^k + \bar{\boldsymbol{n}}^k, \tag{5}$$

we get the likelihood:

$$p(\bar{\boldsymbol{v}}^k|\gamma;\bar{\boldsymbol{\imath}}^k) \propto \exp\Big\{-\frac{1}{2\lambda^2}\sum_{k=1}^{K}\|\bar{\boldsymbol{v}}^k - \boldsymbol{P}\boldsymbol{A}_{\exp\gamma}^{-1}\boldsymbol{P}^\top\boldsymbol{D}\bar{\boldsymbol{\imath}}^k\|^2\Big\}. \tag{6}$$

In EIT applications, especially in medical imaging, conductivity distributions are often homogeneous areas, separated by discontinuities. Therefore, we suggest to use the 2D mesh structure itself to introduce a Markov Random Field (MRF) as prior. Let $v(p)$ be the set of spatial neighbors of the p^{th} element ($v(p)$ contains three elements at most), and let us introduce the $\mathcal{C}^1$ convex Huber penalty function defined by:

$$h_T(t) := \begin{cases} t^2 & \text{if } |t| < T, \\ 2T|t| - T^2 & \text{otherwise} \end{cases}$$

and the prior probability $p(\gamma) = \exp -\Phi_T(\gamma)$, with

$$\Phi_T(\gamma) = \frac{1}{2}\sum_{p=1}^{P} \sum_{q \in v(p)} h_T(\gamma_p - \gamma_q). \qquad (7)$$

$h_T(\gamma_p - \gamma_q)$ favours local smoothness thanks to its near-to-zero parabolic zone, but it also allows discontinuities between pixels thanks to the linear parts [11]. As a result, computing the *maximum a posteriori* amounts to minimizing the following criterion:

$$J(\gamma) = \sum_{k=1}^{K} ||\bar{\boldsymbol{v}}^k - \boldsymbol{P}\boldsymbol{A}_{\exp\boldsymbol{\sigma}}^{-1}\boldsymbol{P}^\top \boldsymbol{D}\bar{\boldsymbol{\imath}}^k||^2 + 2\lambda^2\Phi_T(\gamma). \qquad (8)$$

This criterion is the sum of a non convex likelihood term and a prior convex term. The latter reduces the global non convexity of $J(\gamma)$, so it favours its minimization, even toward a local minimum. Hua *et al.*, who have a white Gaussian regularization term with respect to $\boldsymbol{\sigma}$, use a Newton-Raphson method to minimize their criterion. This second order algorithm is computationaly heavy because it requires Hessian approximation, and looks inadequate because the criterion has not a parabolic shape. We preferred a first order descent method: the conjugate gradient algorithm, which proves simpler and faster.

4 Simulation results

Here, we present the simulated reconstruction results obtained on a « peanut shaped » domain. For the construction of the original conductivity and the data outputs, the domain was discretized into $N = 552$ nodes, $\bar{N} = 100$ border nodes and $P = 1002$ elements. Its conductivity consists of a homogeneous background of 1 $S.m^{-1}$, with two discontinuous areas of 6 $S.m^{-1}$ and 20 $S.m^{-1}$ (*cf.* Fig. 2). For the reconstruction, we took $N = 174$, $P = 296$ and the $\bar{N} = 50$ border points which are chosen as every second point of the finely discretized border. The $K = 7 > P/\bar{N}$ different current and voltage distributions are sampled from the original border distribution in the same way. The current stream inputs are chosen as successive rotation of the same 3-level piecewise constant distribution. Since the problem is very ill-posed, most of the authors considered either free of noise examples [2], or otherwise very weakly degraded. Hua *et al.* worked on real objects and estimated the observation uncertainty measure to 1/2000 (66 dB). To compensate for possible additional modelization uncertainty, we use 40 dB SNR simulations for the voltage observation sets. Simulations are implemented using *Matlab 4.2* software on a *HP 712* workstation.

Like many authors, we choose, as the starting point of our minimization algorithm, the measurable uniform background conductivity of the original distribution. We present here the results obtained with three different reconstruction methods: the first is obtained without regularization, the second with a quadratic (Markov-Gauss) regularization and the third is the proposed method. Parameters λ for the quadratic term and (λ, T) for the Huber function were chosen empirically to get the best qualitative reconstruction results. The threshold T was chosen very small so as to get a quasi-$\mathcal{L}^1$ regularizing effect (which can be detected, on the table below, by the longer convergence time of the algorithm). Besides, the reconstruction quality is not very sensitive to variations of λ (order of magnitude variations are necessary to detect significant influence on the reconstruction result).

In order to compare the quality of the reconstructions $\hat{\gamma}$, we measure the $\mathcal{L}^1$ distance of the estimated distribution to the original. Indeed, the $\mathcal{L}^1$ norm is known to be in good agreement with visual appreciation. More precisely, we use the following discrepancy measure: $\delta_1(\gamma, \hat{\gamma}) = ||\gamma - \hat{\gamma}||_1/||\gamma||_1$, where $\gamma, \hat{\gamma} : \Omega \longrightarrow \mathbb{R}$ are respectively the distributions associated with the vectors $\boldsymbol{\gamma}, \hat{\boldsymbol{\gamma}}$.

Method	λ	T	$\delta_1(\gamma, \hat{\gamma})$	Exec. Time
No regul.	0	∞	0.70	248 *s*
Gaussian	0.012	∞	0.64	244 *s*
Huber	1.5	10^{-5}	0.53	1415 *s*

The table above, as well as reconstructions (*cf.* Figs. 3, 4 and 5), shows that regularization is not only useful but necessary, and that the convex Huber function is relevant for the reconstruction of discontinuities.

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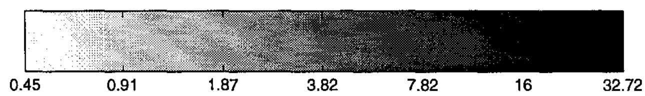

Fig. 1 - Scale used for the conductivity distribution.

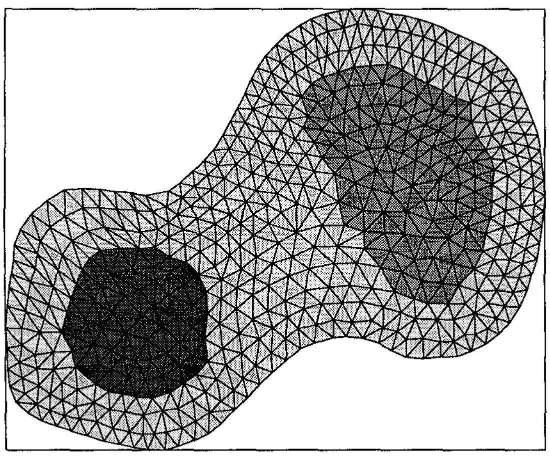

Fig. 2 - Original conductivity distribution.

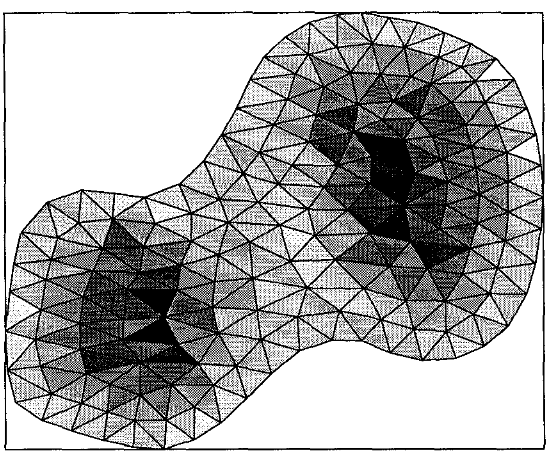

Fig. 3 - Reconstruction without regularization.

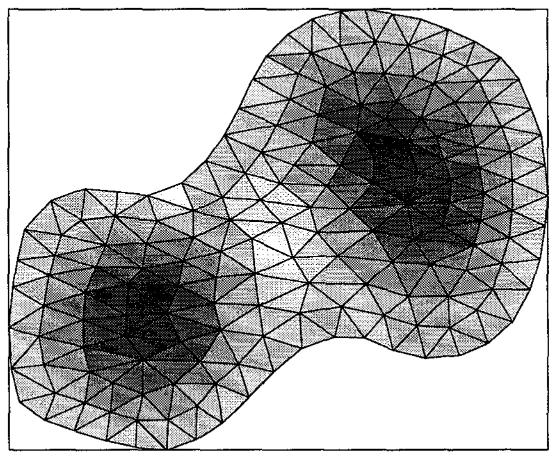

Fig. 4 - Gauss-Markov prior reconstruction.

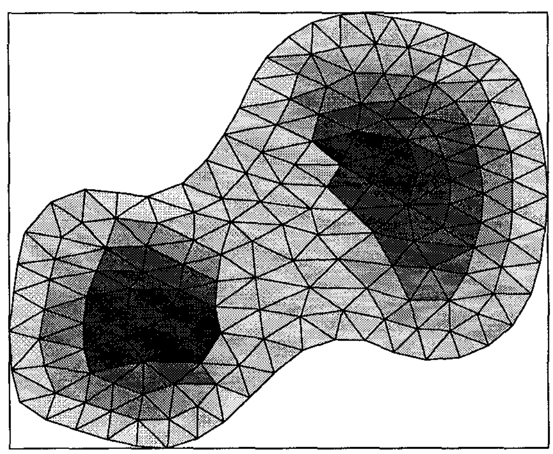

Fig. 5 - Huber-Markov prior reconstruction.