MARKOVIAN HIGH RESOLUTION SPECTRAL ANALYSIS

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ABSTRACT

When short data records are available, spectral analysis is basically an undetermined linear inverse problem. One usually considers the theoretical setting of regularization to solve such ill-posed problems. In this paper, we first show that "nonparametric" and "high resolution" are not incompatible in the field of spectral analysis. To this end, we introduce non quadratic convex penalization functions, like in low level image processing. The spectral amplitudes estimate is then defined as the unique minimizer of a compound convex criterion. An original scheme of regularization to simultaneously retrieve narrow-band and wide-band spectral features is finally proposed.

1. INTRODUCTION

Spectral analysis can be identified as a linear undetermined inverse problem, particularly when a few data are available. The regularization theory [1] is a well-adapted setting to solve such ill-posed problems. Recently, Sacchi *et al.* [2] have proposed a nonparametric method for discrete frequencies spectral estimation. They address this problem as a Fourier synthesis [3]. Indeed, the unknowns are the Fourier coefficients of a sample time series. Since they are more numerous than the data, Sacchi *et al.* rely on the regularization theory to derive a compound criterion composed of two terms: a quadratic and data-driven one and a penalization one. The trade-off between these terms is guaranteed by two hyperparameters. The method proposed in [2] is well adapted to the estimation of line spectra, since they introduce a non convex separable penalization term, which corresponds to an independent Cauchy prior law in a Bayesian framework. As a consequence, such a spectral estimation method can retrieve closely spaced sinusoids.

With the same formulation, we address in this paper the more difficult spectral estimation problem of discrete time compound random process (DTCRP): spectrum of DTCRP is a mixture of narrow-band and wide-band components. Our approach takes advantage of the regularized interpretation of periodograms [4]. We first introduce a strictly convex separable penalization, which is more resolvant than the quadratic one. The corresponding cost function is consequently strictly convex and the discrete frequencies spectral estimate is then defined as the global minimizer of this criterion. A Markovian contribution can also be incorporated to restore smooth spectral components. However, in the presence of DTCRP, this regularization scheme does not allow to retrieve narrow-band and wide-band spectral components simultaneously. To this end, in a second step, we derive a new model relating data to unknowns, called the "mixture model", and we propose a new scheme of regularization. Finally, spectral estimates computed from samples of a process consisting of sinusoids in colored Gaussian noise are presented to illustrate the performances of our techniques.

2. PROBLEM FORMULATION

In our approach, spectral analysis identifies with a Fourier synthesis problem [3]. Therefore, the goal is to estimate the Fourier coefficients of a time series. Consider a sample time complex vector $\boldsymbol{x} = [x_0, \dots, x_{N-1}]^{\mathrm{t}}$. We wish to estimate P spectral components, with $P \gg N$. Let us denote respectively $\nu_p = p/P$ and $X_p = X(\nu_p)$, $p \in \{0, 1, \dots, P-1\} = \mathbb{N}_P$, the equally spaced discrete frequencies and the corresponding spectral amplitudes. Then any sample x_n can be modeled by the inverse discrete Fourier transform of the sample frequency vector $\boldsymbol{X} = [X_0, \dots, X_{P-1}]^{\mathrm{t}} \in \mathbb{C}^P$:

$$x_n = \sum_{p=0}^{P-1} X_p e^{2j\pi\nu_p n},\ n \in \mathbb{N}_N.$$

Denoting $W_{NP} = [w_0^{np}]_{n\in\mathbb{N}_N}^{p\in\mathbb{N}_P}$ the $N \times P$ Fourier matrix where $w_0 = e^{2j\pi/P}$, the previous relations can be written:

$$\boldsymbol{x} = W_{NP}\,\boldsymbol{X}.$$

In practice, $\boldsymbol{x}$ is not available: observed data $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{b}$ are corrupted by modeling and experimental uncertainties. For the sake of simplicity, the noise $\boldsymbol{b}$ is assumed to be a zero-mean, circular, stationary, white and Gaussian vector. Linear relation between the N-sample data vector and the unknown parameters is:

$$\boldsymbol{y} = W_{NP}\,\boldsymbol{X} + \boldsymbol{b}. \tag{1}$$

Since the estimation relies on a small number of data, system (1) is undetermined and therefore can be satisfied by any vector $\widehat{\boldsymbol{X}}$ which minimizes the following least square criterion:

$$\mathcal{Q}(\boldsymbol{X}) = \|\boldsymbol{y} - W_{NP}\,\boldsymbol{X}\|^2.$$

Since $\mathcal{Q}$ is convex only in wide sense, we introduce an additional strictly convex regularization term $\mathcal{R}(\boldsymbol{X})$, to yield a strictly convex compound cost function $\mathcal{J}(\boldsymbol{X})$. We give to $\mathcal{J}$ the following form:

$$\mathcal{J}(\boldsymbol{X}) = \mathcal{Q}(\boldsymbol{X}) + \lambda \mathcal{R}(\boldsymbol{X}),$$

where the regularization parameter $\lambda \geqslant 0$ balances fidelity to the data and fidelity to the penalization term. The estimator of the spectral amplitudes is well-defined as the unique minimizer of this criterion, that is to say:

$$\widehat{\boldsymbol{X}} = \arg\min_{\boldsymbol{X} \in \mathbb{C}^P} \mathcal{J}(\boldsymbol{X}),$$

and the power spectrum estimator deduces as the square modulus of $\widehat{\boldsymbol{X}}$. For example, the special case of the quadratic separable penalization, that is $\mathcal{R}(\boldsymbol{X}) = \boldsymbol{X}^\dagger \boldsymbol{X}$, leads to a low-resolution spectral estimate $\widehat{\boldsymbol{X}}$, which is proportional to the usual periodogram.

3. NON QUADRATIC REGULARIZATION

3.1. Separable penalization

As it is done in image restoration, it can be interesting to replace a quadratic regularization term by a more resolvant one. To this end, Sacchi *et al.* [2] have proposed a penalization term, which corresponds to the choice of a Cauchy prior law in Bayesian estimation:

$$R_0(x) = \ln(1 + x^2/\tau^2),$$

where τ controls the amount of sparseness. The following penalization function encompasses such a possibility:

$$\mathcal{R}(|\boldsymbol{X}|) = \sum_{p=0}^{P-1} R_0(|X_p|). \tag{2}$$

Contrarily to Sacchi *et al.*, we have chosen a strictly convex function for R_0. In order to improve spectral resolution, we assume that $R_0(x)$ grows more slowly than x^2 for large values of x.

Convexity of $R_0(|.|)$ implies the one of $\mathcal{R}(|.|)$, which ensures existence and unicity of a solution in $\mathbb{C}^P$ for the minimization of $\mathcal{J}$. The minimizer is continuous with respect to (w.r.t.) data. This guarantees the well-posedness of the regularized problem [1]. Nevertheless, convexity of R_0 does not imply the one of $R_0(|.|)$. Applied to $g(X) = |X|$ and $f(\rho) = R_0(\rho)$, the following theorem ensures convexity of $R_0(|.|)$ for any function R_0 which is simultaneously convex and increasing on $\mathbb{R}_+$.

Theorem 3.1 *[5, §5]*
Let g be a convex function from $\mathbb{R}^n$ to $\mathbb{R}_+$ and f be a convex function from $\mathbb{R}_+$ to $\mathbb{R}_+$, which is increasing. Then $h = f \circ g$ is convex on $\mathbb{R}^n$.

"L_p" functions ρ^p $(1 \leqslant p < 2)$, commonly used in image reconstruction [6] and "L_{21}" functions like $\sqrt{\beta + \rho^2}$, also used in edge-preserving image restoration [7] satisfy the conditions of theorem 3.1 and increase more slowly than the "L_2" function ρ^2. Therefore, R_0 is chosen among those functions.

We now describe the minimization stage of $\mathcal{J}(\boldsymbol{X})$. Sacchi *et al.* [2] have used a procedure closed to the iteratively reweighted least square (IRLS) algorithm. In this case, the cost function is not necessarily convex, local minima can exist and convergence of IRLS to the global minimizer is not granted. In our case, such a procedure is convergent since we have chosen a strictly convex penalization term. Moreover, it can be shown [8] that the IRLS algorithm identifies with a block-coordinate descent method, such as ARTUR [7], applied to a half-quadratic augmented criterion K_{GR}, which is built with the Geman & Reynolds's duality [9]. K_{GR} satisfies:

$$\forall \boldsymbol{X}, \quad \inf_{\boldsymbol{b}} K_{\mathrm{GR}}(\boldsymbol{X}, \boldsymbol{b}) = \mathcal{J}(\boldsymbol{X}),$$

and has the following form:

$$K_{\mathrm{GR}}(\boldsymbol{X}, \boldsymbol{b}) = \mathcal{Q}(\boldsymbol{X}) + \lambda \sum_{p=0}^{P-1} \left(b_p |X_p|^2 + \psi_0(b_p)\right),$$

where $\boldsymbol{b} = [b_0, \ldots, b_{P-1}]^{\mathrm{t}} \in \mathbb{R}_+^P$ is an auxiliary process, and ψ_0 can be defined from R_0 through convex duality relations (see [9, 7]). In [8], it is shown that K_{GR} is convex in $(\boldsymbol{X}, \boldsymbol{d} = \psi_0(\boldsymbol{b}))$ under some technical conditions including the convexity of R_0. Furthermore, convergence to the global minimum $\widehat{\boldsymbol{X}}$ of $\mathcal{J}$ is established for several kinds of coordinate descent methods [8], provided that $\mathcal{J}$ is C^1 (*i.e.*, continuously differentiable). In the nonsmooth case (*i.e.*, R_0 non differentiable at 0), convergence is also available for an appropriately *under-relaxed* form. Since many versions of deterministic algorithms that operate on K_{GR} converge to $\widehat{\boldsymbol{X}}$, it is not necessary to perform the minimization of $\mathcal{J}$ with a non differentiable optimization algorithm.

3.2. Gibbsian penalization

Separable regularization leads to a significant gain of resolution, but it does not favor retrieval of smooth spectral components (see Fig. 1 (b)-(c)). To strengthen these features, a classical technique consists in appending a Gibbsian term [10, 6] to $\mathcal{R}(\boldsymbol{X})$. If one processes as in image restoration, one can add to R_0 a penalization on the first differences. Since we eventually take interest in restoring the power spectrum rather than the complex spectral amplitudes, we penalize the first differences of the modulus rather than those of the amplitudes:

$$R_1(|\boldsymbol{X}|) = \mu \sum_{p=0}^{P-1} R_1\left(P\left(|X_{p+1}| - |X_p|\right)\right),$$

with the circularity constraint: $X_P = X_0$, since the sought spectrum is 1-periodic. By this way, the smoothness of the power spectrum is ensured. Finally, the total penalization function is:

$$\mathcal{R}(|\boldsymbol{X}|) = \sum_{p=0}^{P-1} [R_0(|X_p|) + \mu R_1(P(|X_{p+1}| - |X_p|))], \tag{3}$$

where functions R_0 and R_1 are assumed to be symetric, convex, increasing and continuously differentiable. To ensure the strict convexity of the penalization function $\mathcal{R}$, it is necessary to give a multivariate extension of theorem 3.1:

Theorem 3.2
Let g be a function from $\mathbb{R}^n$ to $\mathbb{R}_+^m$ and let f be a convex function from $\mathbb{R}_+^m$ to $\mathbb{R}_+$. Suppose moreover that each component g_k of g is convex and that f is a coordinatewise increasing function everywhere in $\mathbb{R}_+^m$, then $h = f \circ g$ is convex on $\mathbb{R}^n$.

We apply this result to $g_k(X_1, X_2) = |X_k|$ for $k \in \{1, 2\}$ and $f(\rho_1, \rho_2) = R_0(\rho_1) + R_0(\rho_2) + 2\mu R_1(P(\rho_1 - \rho_2))$. To conclude that $\mathcal{R}$ is convex on $\mathbb{C}^P$, one needs to verify the following increasing coordinatewise inequality:

$$\forall \rho_1, \rho_2 > 0, \qquad R_0'(\rho_1) \geqslant 2\mu P R_1'(P(\rho_1 - \rho_2)). \qquad (4)$$

A sufficient condition to ensure (4) is

$$\forall \rho_2 > 0, \qquad R_0'(0) \geqslant 2\mu P R_1'(P\rho_2).$$

Since R_0 is supposed to be increasing, we have:

$$\forall\, \rho_1 \geqslant 0,\ R_0'(\rho_1) \geqslant 0.$$

If $R_0'(0) = 0$, R_1 must satisfy:

$$\forall\, \rho_2 > 0, \qquad R_1'(\rho_2) \leqslant 0 \qquad (\mu > 0).$$

This is in contradiction with the above mentioned increasing hypothesis on R_1, except if R_1 is a constant function. Nevertheless, this case is not of any interest, hence we forcibly have $R_0'(0) > 0$. In other words, the functions R_0 which have a linear behaviour in the neighborhood of 0 ensure the increasing coordinatewise condition of f on $\mathbb{R}_+^2$, and then the convexity of $\mathcal{R}$ on $\mathbb{C}^2$. In practice, we have chosen the following function f to ensure the strict convexity of $\mathcal{R}$:

$$f(\rho_1, \rho_2) = \varphi(\rho_1) + \varphi(\rho_2) + 2\mu\psi(P(\rho_1 - \rho_2)),$$

where $\varphi(\rho) = \rho$ or $\varphi(\rho) = \sqrt{\beta + (\rho+\gamma)^2}$ and $\psi(\rho) = \sqrt{\alpha + \rho^2}$. Parameters of f must satisfy $2\mu P < 1$ and $2\mu P < \gamma/\sqrt{\beta+\gamma^2}$ respectively.

Finally, the strict convexity of the global criterion $\mathcal{J}$ is obtained to the detriment of its differentiability at the origin. Therefore, we emphasize here the fact that usual efficient algorithms of minimization, like gradient methods, do not apply.

If we add to the L_1 potential a strictly convex differentiable term in R_0, the global expression of $\mathcal{J}$ allows a successful application of a coordinate descent method according to [11, theorem 1.4, p. 73]. However, convergence of this algorithm is very slow. To simplify the minimization task, we propose a fast alternative based on *half-quadratic regularization* [12] and alternate deterministic minimizations [7]. Unfortunately, we have not presently brought guaranties of convergence towards the global minimizer. Nevertheless, we give the principle of our algorithm.

Introduction of an auxiliary process for the Markovian term R_1 as in Subsection 3.1 yields a quadratic term in the differences of spectral modulus, but not in those of spectral amplitudes. In [13], it is shown that a multivariate extension of Geman & Yang's *half-quadratic regularization* [12] can solve this problem, provided that two auxiliary variables are matched to each spectral amplitude. Geman & Yang exploit Legendre pairs to introduce a new objective function. Since Legendre pairs [5, §26] are not well defined for non smooth functions, such a procedure does not apply to R_0, whereas the Geman & Reynolds's duality can apply. The corresponding augmented criterion K_{GRY} satisfies:

$$\forall \boldsymbol{X}, \qquad \inf_{\boldsymbol{l}^{(+,-)}} K_{\text{GRY}}(\boldsymbol{X}, \boldsymbol{b}, \boldsymbol{l}^{(+,-)}) = K_{\text{GR}}(\boldsymbol{X}, \boldsymbol{b}),$$

and has the following form:

$$K_{\text{GRY}}(\boldsymbol{X}, \boldsymbol{b}, \boldsymbol{l}^{(+,-)}) = K_{\text{GR}}(\boldsymbol{X}, \boldsymbol{b}) + \mu \sum_{p=0}^{P-1} \left(\frac{1}{2}\left[\left|X_p - l_p^+\right|^2 + \left|X_p - l_p^-\right|^2 \right] + \zeta_1\left(\left|l_p^+\right| - \left|l_p^-\right|\right) \right),$$

where $(\boldsymbol{l}^+, \boldsymbol{l}^-) \in (\mathbb{C}^P)^2$ are auxiliary processes, and ζ_1 can be defined from R_1 through Legendre transform [12]. To perform the minimization of $\mathcal{J}$, we have used a strategy based on alternate minimizations over $\boldsymbol{X}$, $\boldsymbol{b}$ and $\boldsymbol{l}^{(+,-)}$. Simulations with a Markovian penalization provide quite accurate power spectrum estimates for the broad-band portion of DTCRP. By contrast, line spectra are not well retrieved (see Fig. 1 (d)). We now focus on the "mixture model", which takes into account both broad-band and narrow-band parts of the DTCRP.

4. THE MIXTURE MODEL

We give a new formulation of spectral estimation for the DTCRP. Since the DTCRP has spectral peaks but also smooth spectral components, it seems to be relevant to include these features in the model relating data to unknowns. To this end, the unknown spectrum $\boldsymbol{X}$ is modeled as the sum of two terms: one for the narrow-band portion $\boldsymbol{X}^{\text{n}}$, and another one for the wide-band component $\boldsymbol{X}^{\text{w}}$. This construction leads to the name of "mixture model". Relation (1) becomes:

$$\boldsymbol{y} = W_{NP}\,(\boldsymbol{X}^{\text{n}} + \boldsymbol{X}^{\text{w}}) + \boldsymbol{b},$$

and the least square criterion $\mathcal{Q}_{\text{M}}$ is written:

$$\mathcal{Q}_{\text{M}}(\boldsymbol{X}^{\text{n}}, \boldsymbol{X}^{\text{w}}) = \|\boldsymbol{y} - W_{NP}\,(\boldsymbol{X}^{\text{n}} + \boldsymbol{X}^{\text{w}})\|^2.$$

Specific penalization terms are introduced for both $\boldsymbol{X}^{\text{n}}$ and $\boldsymbol{X}^{\text{b}}$. For the narrow-band part $\boldsymbol{X}^{\text{n}}$, we consider as in (2) a separable differentiable penalization function $\mathcal{R}^{\text{n}}$. In the same way, for the wide-band portion $\boldsymbol{X}^{\text{w}}$, we choose as in (3) a Markovian strictly convex regularization term $\mathcal{R}^{\text{w}}$. The total "mixture" penalization function combines $\mathcal{R}^{\text{n}}$ and $\mathcal{R}^{\text{w}}$:

$$\mathcal{R}_{\text{M}}(\boldsymbol{X}^{\text{n}}, \boldsymbol{X}^{\text{w}}) = \lambda_{\text{n}}\mathcal{R}^{\text{n}}(\boldsymbol{X}^{\text{n}}) + \lambda_{\text{w}}\mathcal{R}^{\text{w}}(\boldsymbol{X}^{\text{w}}),$$

where $\lambda_{\text{n}}, \lambda_{\text{w}} > 0$ are the hyperparameters. To strengthen the restoration of line spectra in $\boldsymbol{X}^{\text{n}}$, we have to choose $\lambda_{\text{n}} > \lambda_{\text{w}}$. Finally, the global "mixture" criterion $\mathcal{J}_{\text{M}}$ is:

$$\mathcal{J}_{\text{M}}(\boldsymbol{X}^{\text{n}}, \boldsymbol{X}^{\text{w}}) = \mathcal{Q}_{\text{M}}(\boldsymbol{X}^{\text{n}}, \boldsymbol{X}^{\text{w}}) + \mathcal{R}_{\text{M}}(\boldsymbol{X}^{\text{n}}, \boldsymbol{X}^{\text{w}}).$$

The uniqueness of the minimizer $(\boldsymbol{X}^{\text{n}}, \boldsymbol{X}^{\text{w}})$ of $\mathcal{J}_{\text{M}}$ is ensured, because pointwise addition of a convex function with another strictly convex one is strictly convex [5].

An extension of the previous algorithm of optimization have been used to carry out the minimization of $\mathcal{J}$ w.r.t. $\boldsymbol{X}^{\text{n}}$ and $\boldsymbol{X}^{\text{w}}$. This procedure now requires a more expensive computational burden than the one of Section 3, but the accuracy of the results justifies such a contribution.

5. SIMULATION RESULTS

We illustrate the performances of our spectral estimation method, using data similar to the Kay-Marple [14] ones. Each spectral estimate is based on a sequence of 64 complex sample points extracted from a process consisting of three sinusoids and a Gaussian colored noise process. The true power spectral density (PSD) is shown in Fig. 1 (f). The three sinusoids are at fractional frequencies of -0.3, -0.22 and -0.21 and have SNR's of 0, +20 and +20 dB, where SNR is defined as the ratio of sinusoid power to the total power in the passband noise process. The Gaussian noise process passband is zero-mean with standard deviation $\sigma = 0.05$. This particular signal is selected to compare our spectral analysis techniques with both narrow-band and wide-band processes.

The PSD estimates, pictured in Fig. 1, are obtained from the minimization of penalized criteria. In each case, a common hyperparameter to each regularization term is chosen such that $\|\widehat{X}\|^2 = \|y\|^2$. The other parameters are selected empirically. Each spectral estimate depicted here is computed using 256 frequency samples. A zero-padded periodogram is shown in Fig. 1 (a). The nominal resolution of a 64-point sequence is 0.015, so that the sinusoids at -0.22 and -0.21 are closer than the resolution width. Two PSD estimates obtained with different convex separable penalizations are depicted in Fig. 1 (b)-(c). The second one is very similar to the spectral estimate computed with the Cauchy-Gauss model in [2]. Fig. 1 (d) shows the PSD estimate performed by our first convex Markovian model (3). The broad-band response of such a Markovian spectrum is very accurate. Unfortunately, it is unable to resolve the two close sinusoids. The Markovian "mixture model" yields the PSD estimate shown in Fig. 1 (e). It gives more accurate results than the other known nonparametric methods: the three sinusoid components have sharp responses at the sinusoid frequencies, and the broad-band response is very similar to that of the single Markovian estimate.

6. CONCLUSION

We have examined the problem of nonparametric spectral estimation for the DTCRP. We have shown that separable spectral estimates based on convex penalized criteria provide a quite accurate narrow-band response. To improve the quality of the wide-band response, we have introduced a Markovian penalization in the criterion. In this case, the closely spaced sinusoids are not resolved whereas the broad-band component is well retrieved. To avoid such a disadvantage, we have proposed an original model and an adapted regularization function. Since each estimate is obtained via the minimization of a convex criterion, it is computed by an optimization procedure. We have used a fast algorithm, whose convergence to the global minimizer is only granted in the separable case.

7. REFERENCES

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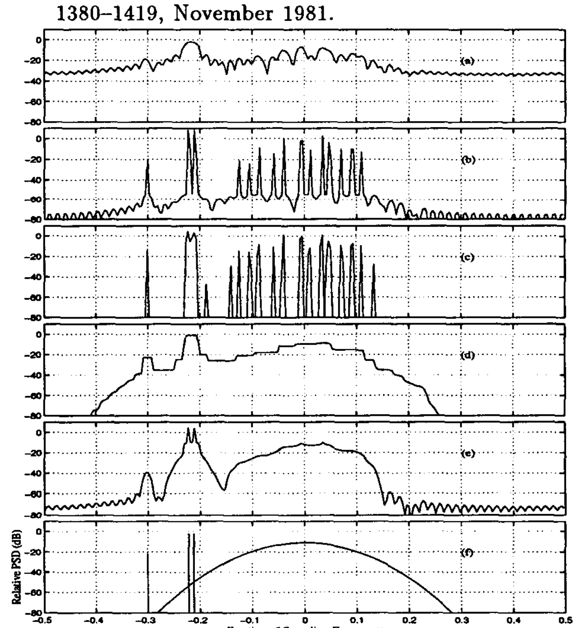

Figure 1: Zero-padded periodogram (a) and non quadratic extensions: (b) separable penalization with L_{21} potential ; (c) separable penalization with L_1 potential ; (d) Markovian penalization on the modulus first differences ; (e) Markovian penalization with the "mixture" model ; (f) True PSD.