

Stationary Markov Random Fields on a Finite Rectangular Lattice

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Abstract—This paper provides a complete characterization of stationary Markov random fields on a finite rectangular (non-toroidal) lattice in the basic case of a second-order neighborhood system. Equivalently, it characterizes stationary Markov fields on \mathbb{Z}^2 whose restrictions to finite rectangular subsets are still Markovian (i.e., even on the boundaries). Until now, Pickard random fields formed the only known class of such fields. First, we derive a necessary and sufficient condition for Markov random fields on a finite lattice to be stationary. It is shown that their joint distribution factors in terms of the marginal distribution on a generic (2×2) cell which must fulfill some consistency constraints. Second, we solve the consistency constraints and provide a complete characterization of such measures in three cases. Symmetric measures and Gaussian measures are shown to necessarily belong to the Pickard class, whereas binary measures belong either to the Pickard class, or to a new nontrivial class which is further studied. In particular, the corresponding fields admit a simple parameterization and may be simulated in a simple, although nonunilateral manner.

Index Terms—Markov random fields, Pickard random fields, stationarity, unilaterality.

I. INTRODUCTION

IN recent years, Markov random fields (MRF's) have been the subject of a renewed interest in the image processing community [1]–[3]. MRF's form a wide class of stochastic processes that model qualitative information about scenes and textures such as piecewise homogeneity in a somewhat natural way.

This paper investigates the compatibility of stationarity and Markovianity for random fields designed on finite rectangular subsets of \mathbb{Z}^2 . Toroidal lattices enable simple design of such stationary MRF's, but for image-processing purposes the resulting neighborhood system induces undesirable interactions between opposite boundaries. Therefore, they are excluded from this discussion: the neighborhood system considered here is basically inherited from the Euclidean distance between sites.

Stationarity usually refers to processes indexed on an infinite set and means that the marginal probability of any subset of

sites remains invariant under any translation. This definition can readily be extended to the case of cyclic boundaries (like the toroidal lattice). However, in order to remain consistent with the remark stated in the previous paragraph, stationarity strictly refers to the translations that keep the given subset inside the index set, so that the marginal of a stationary field on \mathbb{Z}^2 is stationary in our sense.

By the Hammersley–Clifford theorem, MRF's and Gibbs random fields (GRF's) on a finite lattice are equivalent under the positivity condition, and the most widespread approach for practical manipulation of MRF's is undoubtedly the specification of *Gibbs potentials*. However, manipulation of marginals on subsets by means of Gibbs potentials is particularly uneasy [4]. In particular, no general condition of stationarity is known for GRF's on a regular finite lattice. Yet, stationary MRF's may be designed on infinite lattices through specification of conditional probabilities [5], [6]. By definition of stationarity, their marginal probability distributions on finite sublattices are stationary, but the Markov property is generally lost on the boundaries of the sublattice. One might resort to manipulating the resulting “almost-Markov” stationary distributions, but here again, the general derivation of such probability distributions is an open problem. One may finally wonder if Markovianity and stationarity are compatible on finite lattices. Our study provides precise answers to this question.

Presently, the only acknowledged class of nontrivial stationary Markov fields on finite plane rectangular lattices is due to Pickard [7], [8]. The so-called “Pickard random fields” (PRF's) exhibit some very unusual properties. In particular, they are unilateral (causal), so that they can be simulated recursively in a straightforward manner, and each row and column in the lattice is a stationary Markov chain. Such properties are statistically appealing and they were extensively used by Devijver and Dekessel [9] to derive fast unsupervised segmentation methods. However, since the pioneering work of Pickard, no other class of stationary MRF's on regular lattices has been exhibited, except for the class of “Markov–Bernoulli” random fields [10], which were specifically designed for modeling stratified media. Actually, the work of Pickard left some unresolved issues about the generality of PRF's: do they form the only class of stationary second-order MRF's on finite rectangular lattices? What is the role of unilaterality, which is imposed as a prerequisite in his work? To our knowledge, the only attempt to answer these questions is due to Goutsias [11], who considers the class of GRF's with *homogeneous potentials* and derives necessary and sufficient conditions on the potentials for these fields to be stationary. Unfortunately,

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these conditions are implicit and they are too intricate to be put in closed form. The problem addressed here is the characterization of stationary MRF's on finite rectangular lattices. This study can be seen as an extension of the work of Goutsias [11]. First, homogeneity of the Gibbs potentials is not imposed as a prerequisite, the only requirement being stationarity. Second, the necessary and sufficient conditions are solved in three cases of general interest; this can be viewed as the main contribution of this paper. For the sake of simplicity of the presentation, the emphasis is placed on *second-order* MRF's on \mathbb{Z}^2 . However, we also indicate in the conclusion how some of the results could be extended to higher dimensions and/or to higher order neighborhoods.

It has been known for a long time that stationary first-order Markov (bilateral) processes indexed on \mathbb{Z} are equivalent to stationary first-order Markov chains (MC's) under the positivity assumption (e.g., see Dobrushin [5]). Consequently, the marginal distribution of such processes on a finite subset remains Markovian even on the boundaries. This property does not generally hold for stationary MRF's on \mathbb{Z}^2 . The work presented here also yields a characterization of stationary, positive second-order MRF's on \mathbb{Z}^2 such that the marginals on all finite rectangles remain Markovian.

The paper is organized as follows: Section II is devoted to a formal definition of the problem and to basic properties of Markov chains (MC's), MRF's, and PRF's. Section III establishes useful equivalent definitions of stationary second-order MRF's. In particular, it is shown that the probability distributions of these fields factor in terms of their marginals on a generic 2×2 cell. These marginals satisfy some nontrivial constraints expressed in an implicit form. Since these constraints do not lend themselves to a comprehensive treatment, three cases of general interest are investigated and solved in Section IV, i.e., symmetric, Gaussian, and binary distributions; only the last of which yields non-Pickard solutions. To our knowledge, this class of non-Pickard stationary binary MRF's was never studied before. A complete parameterization is provided, as well as a simple sampling scheme which is recursive although not unilateral. Section V deals with extension of stationary second-order MRF's on finite lattices to \mathbb{Z}^2 . Section VI presents some concluding remarks and some open issues.

II. PROBLEM FORMULATION AND DEFINITIONS

A. Definitions and Basic Properties

In this section, we formally define and state basic properties of MRF's, MC's, and stationary measures.¹ Whenever unambiguous, measure $P(X = x|Y = y)$ will be noted $P(x|y)$, where X (resp., Y) denote any random variable and x (resp., y) one of its realizations. Let Λ be any nonempty finite subset of \mathbb{Z}^p , and let the set of q -order neighbors of any site $s \in \Lambda$ be defined as

$$N_s \triangleq \{r \in \mathbb{Z}^p, \|r-s\|_2^2 \leq q\} \cap \Lambda^{(s)}, \quad \text{where } \Lambda^{(s)} \triangleq \Lambda \setminus \{s\}.$$

¹Throughout the paper "measure" stands for "probability measure."

Definition 1: An MRF $_q^p$ \mathbf{X} on Λ is a collection of random variables $(X_s, s \in \Lambda)$, sampled on a common finite² state space denoted E , that satisfies the following two properties:

$$\forall \mathbf{x} \in E^\Lambda, \quad P(\mathbf{X} = \mathbf{x}) > 0 \quad (1)$$

$$\forall s \in \Lambda, \quad P(x_s|x_r, r \in \Lambda^{(s)}) = P(x_s|x_r, r \in N_s). \quad (2)$$

Assumption (2) defines the Markovian property whereas the "positivity" assumption (1) is a technical restriction required in order for the Hammersley–Clifford theorem [12] to be in force. Note that (2) is equivalent to

$$\forall s \in \Lambda, \forall A \subset \Lambda,$$

$$N_s \subset A \Rightarrow P(x_s|x_r, r \in A^{(s)}) = P(x_s|x_r, r \in N_s). \quad (3)$$

The following study is restricted to the set of MRF $_2^2$'s, i.e., the MRF's defined on finite subsets of \mathbb{Z}^2 that admit interactions among the eight closest neighbors only. Let Λ_{mn} denote the finite *rectangular* lattice

$$\{(k, l) \in \mathbb{Z}^2, 1 \leq k \leq m \text{ and } 1 \leq l \leq n\}.$$

Throughout the paper, we assume that $\Lambda = \Lambda_{MN}$, $M, N > 2$. Let A be any subset of Λ , then \mathbf{X}_A denotes the random field $\{X_{mn}, (m, n) \in A\}$. The random vector \mathbf{X}_m^- (resp., \mathbf{X}_n^+) denotes the collection (X_{m1}, \dots, X_{mN}) (resp., (X_{1n}, \dots, X_{Mn})) and will be referred to as the m th row (resp., n th column) of \mathbf{X} . The same conventions apply to any realization \mathbf{x} of \mathbf{X} .

Specification of an MRF $_2^2$ may be carried out by means of Gibbs potentials: for all $(m, n) \in \Lambda_{M-1, N-1}$ let $V_{mn} \binom{a \ b}{c \ d}$ be a real-valued function defined on E^4 (the so-called Gibbs potential). Then it is well known [2], [12] that the following expression defines an MRF $_2^2$:

$$P(\mathbf{X} = \mathbf{x}) = \frac{1}{Z} \exp \left(\sum_{m=1}^{M-1} \sum_{n=1}^{N-1} V_{mn} \binom{x_{mn} \quad x_{m, n+1}}{x_{m+1, n} \quad x_{m+1, n+1}} \right) \quad (4)$$

where

$$Z = \sum_{\mathbf{x} \in E^\Lambda} \exp \left(\sum_{m=1}^{M-1} \sum_{n=1}^{N-1} V_{mn} \binom{x_{mn} \quad x_{m, n+1}}{x_{m+1, n} \quad x_{m+1, n+1}} \right)$$

is classically referred to as *the partition function*. Reciprocally, the Hammersley–Clifford theorem [12] states that any MRF $_2^2$ admits such a representation.

As explained later in Section II-B, another way to specify Markov measures is to use MC techniques. We stress the fact that the following definitions for MC's and stationarity refer to processes indexed by a *finite* set, whereas the usual definitions involve infinite index sets.

²The finiteness assumption can be relaxed provided the Hammersley–Clifford theorem applies; therefore, all derivations and definitions given hereafter may be generalized to infinite state spaces under appropriate assumptions [12], which are fulfilled in the Gaussian case.

Definition 2: An MC \mathbf{Y} is a collection of RV's (Y_1, \dots, Y_T) such that

$$\forall (y_1, \dots, y_t), \quad P_t(y_t|y_1, \dots, y_{t-1}) = P_t(y_t|y_{t-1}). \quad (5)$$

Moreover, when (Y_1, \dots, Y_T) sample a common state space F and $P(Y_t = y_2|Y_{t-1} = y_1)$ is independent from t , \mathbf{Y} is referred to as a *homogeneous MC*.

It is easy to check that a *positive MC* (MC^+) \mathbf{Y} , i.e., an MC such that $\forall \mathbf{y} \in F^T, P(\mathbf{y}) > 0$, satisfies a ‘‘bilateral’’ property

$$P(y_t|y_1, \dots, y_{t-1}, y_{t+1}, \dots, y_T) = P(y_t|y_{t-1}, y_{t+1}). \quad (6)$$

Therefore, an MC^+ is also an MRF_1^+ . Reciprocally, it may be easily proved, using the Hammersley–Clifford theorem, that an MRF_1^+ is also an MC^+ . Note that the equivalence between bilateral (MRF_1^+) and unilateral (MC^+) representations extends to processes indexed on \mathbb{Z} under the positivity assumption [5]. On the other hand, nonpositive counterexamples may be found in the literature [13], [14].

Definition 3: A random field \mathbf{X} on Λ_{MN} is a positive vector Markov chain by rows (VMC^-) (resp., by columns (VMC^+)) if $(\mathbf{X}_1^-, \mathbf{X}_2^-, \dots, \mathbf{X}_M^-)$ (resp., $(\mathbf{X}_1^+, \mathbf{X}_2^+, \dots, \mathbf{X}_N^+)$) is an MC^+ .

Definition 4: Let \mathcal{T} be the set of translations on \mathbb{Z}^p . Then the stochastic process $\mathbf{X} = (X_s, s \in \Lambda)$ is stationary if $\forall \tau \in \mathcal{T}, \forall A \subset \Lambda$ such that $\tau(A) \subset \Lambda$, and

$$\forall \mathbf{x}_A \in F^A, \quad P(\mathbf{X}_{\tau(A)} = \mathbf{x}_A) = P(\mathbf{X}_A = \mathbf{x}_A).$$

For instance, a positive measure $\pi(u, v)$ on F^2 is stationary if it satisfies, for any $u \in F$,

$$\sum_{v \in F} \pi(u, v) = \sum_{v \in F} \pi(v, u).$$

Then, both marginals are identical and denoted $\pi(u)$, using a common notational abuse.

We have the following obvious characterization for stationary MC^+ :

*Proposition 1:*³ An MC^+ \mathbf{Y} is stationary if and only if there exists a positive and stationary measure on F^2 , $\pi(u, v)$, such that

$$P(\mathbf{y}) = \frac{\pi(y_1, y_2) \cdots \pi(y_{T-1}, y_T)}{\pi(y_2) \cdots \pi(y_{T-1})}. \quad (7)$$

Consequently, the distribution of a stationary MC^+ is fully characterized by the joint distribution of its first two states; such distributions will be referred to as *generic measures*.

Symmetrical expressions such as (7) are seldom used in the literature, where one rather invokes the following unilateral factorization:

$$P(\mathbf{y}) = P(y_1)P(y_2|y_1) \cdots P(y_T|y_{T-1}).$$

One of the aims of this paper is to generalize (7) to stationary MRF's.

³We will make a distinction between Propositions, which sum up known but seldom reported results, and Theorems which are presumed original.

We finally introduce the following implicit notation for the measure on any rectangular ($m \times n$) sublattice of Λ :

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \triangleq P(\mathbf{X}_{\Lambda_{mn}} = \mathbf{x}_{\Lambda_{mn}}).$$

In the same way, conditional distributions will be noted as

$$\begin{bmatrix} x_{11} & \cdots & x_{1l} & \cdots & x_{1n} \\ \vdots & & & & \vdots \\ x_{k1} & \cdots & \boxed{x_{kl}} & \cdots & x_{kn} \\ \vdots & & & & \vdots \\ x_{m1} & \cdots & x_{ml} & \cdots & x_{mn} \end{bmatrix} \triangleq P(X_{kl} = x_{kl} | X_{ij} = x_{ij}, (i, j) \in \Lambda_{mn}^{(kl)}).$$

B. Pickard Random Fields

The only known class of nontrivial stationary MRF's on a finite rectangular (nontoroidal) lattice is due to Pickard [7], [8]. PRF's possess some very specific properties. In particular, the restriction of a PRF to any rectangular sublattice remains a PRF and the marginal distribution of a row or a column is Markovian. Actually, we show in Section III that these properties hold for all stationary MRF_2^+ 's. Unilaterality is another salient feature of PRF's. Pickard considers a measure $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a generic cell, such that B and C are independent conditionally to A (hereafter denoted by $B \perp C|A$). Then he defines the measure $P(\mathbf{x})$ of a unilateral process \mathbf{X} on Λ as follows:

$$P(\mathbf{x}) = \sigma(x_{11}) \prod_{m=2}^M \sigma(x_{m1}|x_{m-1,1}) \cdot \prod_{n=2}^N \sigma(x_{1n}|x_{1,n-1}) \prod_{m=2}^M \prod_{n=2}^N \sigma \begin{pmatrix} x_{m-1,n-1} & x_{m-1,n} \\ x_{m,n-1} & \boxed{x_{mn}} \end{pmatrix}. \quad (8)$$

This defines a homogeneous VMC by rows and by columns, where $(\mathbf{X}_1^-, \mathbf{X}_2^-)$ and $(\mathbf{X}_1^+, \mathbf{X}_2^+)$ form also homogeneous VMC's by columns and by rows, respectively [8]. Pickard investigates additional conditions to be imposed on σ in order to enforce stationarity of \mathbf{X} . In particular, he provides the following sufficient condition.

Proposition 2: Assume $B \perp C|A$. A unilateral process defined by (8) is stationary if

$$(B \perp C|D) \text{ or } (A \perp D|B \text{ and } A \perp D|C), \quad (9)$$

This condition also proves to be necessary in the binary case [8]. However, the role of the basic constraint $B \perp C|A$, remains unclear [8, Remark 2, p. 659]. We will show in Section IV that such a condition arises from global constraints on the random field.

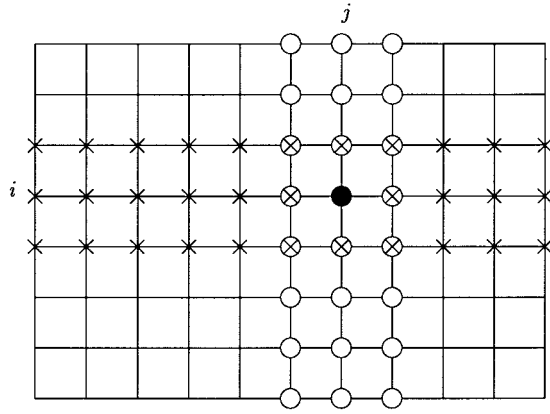


Fig. 1. Illustration of the proof of the sufficient condition in Theorem 1. When \mathbf{X} is a VMC^\dagger (resp., VMC^-), \circ (resp., \times), depicts the set of neighboring sites of site (i, j) . When \mathbf{X} is both a VMC^\dagger and VMC^- , the set of neighbors of site (i, j) is the intersection of both previous neighboring sets, i.e., a second-order neighborhood.

III. EQUIVALENT CHARACTERIZATIONS OF STATIONARY MRF'S

In this section we derive equivalent characterizations of a stationary MRF_2^2 , first in terms of MC's by rows and columns, then in terms of its marginal $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on a generic cell. In the sequel, this marginal will be referred to as the *generic measure*. Most of the results presented here rely on MC properties by rows and by columns. The main results presented in this section rely on the following equivalence between MRF_2^2 and VMC properties.

Theorem 1: \mathbf{X} is an MRF_2^2 iff \mathbf{X} is both a VMC^- and a VMC^\dagger .

The necessary condition is well known (see [15]) and derives from the Hammersley–Clifford theorem. We are not aware of any previous statement of the sufficient condition. Its proof derives from purely Markovian considerations. Since it is straightforward, it is only sketched in Fig. 1.

Applying Theorem 1 in a stationary context yields

Corollary 1: \mathbf{X} is a stationary MRF_2^2 iff \mathbf{X} is both a stationary VMC^- and a stationary VMC^\dagger .

Proof: The Markovian part of the Corollary derives from Theorem 1. The necessary condition is true by definition. The proof of the sufficient condition uses the same arguments as [11, Corollary 3, p. 1242]. \square

An important property of MC's is that the marginal distribution of any subset of sites remains Markovian. The two-dimensional (2-D) counterpart consists of checking whether the marginal distribution \mathbf{X}_A of any rectangular sublattice A of Λ remains Markovian.

As shown by the following counterexample, the property does not hold for any MRF_2^2 .

Counterexample 1: Let \mathbf{X} be the Gaussian (nonstationary) MRF_2^2 defined on Λ by the Gibbs potentials

$$V(\mathbf{x}) = \sum_{s \in \Lambda} x_s^2 + 2\alpha \sum_{s \sim r} x_s x_r \quad (10)$$

where “ $s \sim r$ ” stands for “ r and s are first-order neighbors” and $0 < |\alpha| < 1/2$. As shown in Appendix A, $\mathbf{X}_{\Lambda_{M-1, N}}$ is not an MRF_2^2 : the neighborhood of each site on the last row of $\mathbf{X}_{\Lambda_{M-1, N}}$ extends to the entire row.

Conversely, the following theorem states that this does not occur for a stationary MRF_2^2 .

Theorem 2: Let \mathbf{X} be a stationary MRF_2^2 on Λ and A be any rectangular sublattice of Λ , then \mathbf{X}_A is a stationary MRF_2^2 on A .

By definition of stationarity, if \mathbf{X} is a stationary process then \mathbf{X}_A is also stationary and (3) shows that the Markov property is preserved at any site inside A . Therefore, the issue is to prove that stationarity is sufficient to preserve Markovianity at the boundaries of A . Note that stationarity is not necessary: let \mathbf{Y} be the result of a pointwise inhomogeneous one-to-one transform of a stationary MRF_2^2 \mathbf{X} , then \mathbf{Y} is a nonstationary MRF_2^2 and, by Theorem 2, \mathbf{Y}_A remains an MRF_2^2 . Before turning to the proof itself, let us examine two important consequences of this theorem.

- 1) Since the distribution of a stationary MC is entirely determined by the joint distribution of two contiguous states, the distribution of \mathbf{X} only depends on the distribution of $(\mathbf{X}_1^-, \mathbf{X}_2^-)$ (or $(\mathbf{X}_1^\dagger, \mathbf{X}_2^\dagger)$). By Theorem 2, $(\mathbf{X}_1^-, \mathbf{X}_2^-)$ is a stationary MRF_2^2 , and by Corollary 1, $(\mathbf{X}_1^-, \mathbf{X}_2^-)$ defines a stationary VMC^\dagger . It follows that $P(\mathbf{x}_1^-, \mathbf{x}_2^-)$ (and, therefore, $P(\mathbf{x})$) only depends on the measure of its first two columns, i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note at this point the remarkable analogy with the “ P_{mn} construction” of Pickard [7], [8].
- 2) A may be any row or column of Λ (this case is dealt with in Lemma 2), thus the marginal distribution of any row or column of a stationary MRF_2^2 defines a stationary MC. From the work of Pickard, it is already known that this property holds for PRF's.

Theorem 2 results from the following two lemmas.

Lemma 1: Let M' and N' denote two integers such that $2 \leq M' \leq M$ and $2 \leq N' \leq N$. If \mathbf{X} is a stationary MRF_2^2 , then $\mathbf{X}_{\Lambda_{M'N'}}$ and $\mathbf{X}_{\Lambda_{MN'}}$ are stationary MRF_2^2 's.

The proof of Lemma 1 is straightforward from Fig. 2.

Lemma 2: If \mathbf{X} is a stationary MRF_2^2 then the marginal distribution of its rows and columns define two stationary MC's.

The proof makes use of Lemma 1 and is given in Appendix B. The proof of Theorem 2 is now straightforward:

Proof: By stationarity of \mathbf{X} , we may consider the case $A = \Lambda_{M'N'}$ without loss of generality. The cases $M' = 1$ or $N' = 1$ are covered by Lemma 2. When $2 \leq M' \leq M$ and $2 \leq N' \leq N$, the result derives from the inclusions $\Lambda_{M'N'} \subset \Lambda_{M'N} \subset \Lambda_{MN}$ and successive applications of Lemma 1. \square

Finally, Theorem 2 forms the basis for further characterization of a stationary MRF_2^2 in terms of its distribution

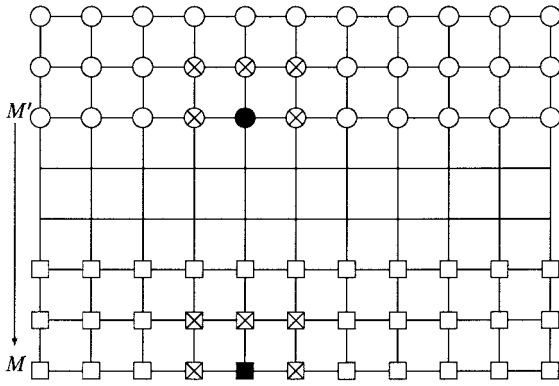


Fig. 2. Illustration of the proof of Lemma 1: by stationarity of \mathbf{X} , the conditioning of the site denoted by \bullet w.r.t. sites denoted by \circ is identical to the conditioning of the site denoted by \blacksquare w.r.t. sites denoted by \square . Since the set of \square contains the second-order neighborhood (depicted by \times) of \blacksquare , conditioning w.r.t. \square is equivalent to conditioning w.r.t. \times . By stationarity, conditioning of \bullet w.r.t. \circ is equivalent to conditioning w.r.t. \times .

$P(\mathbf{x})$. The following theorem provides a 2-D counterpart of decomposition (7).

Theorem 3: An MRF $_2^2$ \mathbf{X} is stationary on the finite rectangular lattice Λ if and only if $P(\mathbf{x})$ admits the following expression in terms of the generic measure as shown in (11) at the bottom of this page, where the generic measure fulfills the following consistency conditions:

$$\forall \mathbf{u} \in E^N, \frac{\prod_{n=1}^{N-1} [u_n \ u_{n+1}]}{\prod_{n=2}^{N-1} [u_n]} = \sum_{\mathbf{v}} \frac{\prod_{n=1}^{N-1} [u_n \ u_{n+1}]}{\prod_{n=2}^{N-1} [v_n]} \quad (12a)$$

$$= \sum_{\mathbf{v}} \frac{\prod_{n=1}^{N-1} [v_n \ v_{n+1}]}{\prod_{n=2}^{N-1} [u_n]} \quad (12b)$$

$$\forall \mathbf{u} \in E^M, \frac{\prod_{m=1}^{M-1} [u_m \ u_{m+1}]}{\prod_{m=2}^{M-1} [u_m]} = \sum_{\mathbf{v}} \frac{\prod_{m=1}^{M-1} [u_m \ v_m]}{\prod_{m=2}^{M-1} [u_m \ v_m]} \quad (13a)$$

$$= \sum_{\mathbf{v}} \frac{\prod_{m=1}^{M-1} [v_m \ u_m]}{\prod_{m=2}^{M-1} [v_m \ u_m]} \quad (13b)$$

Proof: See Appendix C. □

In the sequel, Π_{22} denotes the set of positive and stationary measures indexed on Λ_{22} , and $\Pi_{MN}(M, N > 2)$ is the subset of Π_{22} such that consistency constraints (12) and (13) be satisfied. In the one-dimensional (1-D) case, Proposition 1 reads $\Pi_{1N} = \Pi_{12}$ for all $N > 1$. The 2-D counterpart is more complex, since $\Pi_{MN} = \Pi_{2N} \cap \Pi_{M2}$ (this is a direct consequence of Theorem 3), but $\Pi_{2N} \not\subseteq \Pi_{22}$, as shown by the following counterexample:

Counterexample 2: Consider the subclass of Π_{22} of first-order isotropic Gaussian generic measures defined through the potential

$$V(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 2\alpha(ab + cd + ac + bd) \quad (14)$$

for some $0 < |\alpha| < 1/2$. Symmetry enforces stationarity on Λ_{22} , but, as shown in Appendix D, such measures do not belong to Π_{2N} .

Note that when Pickard constraint $B \perp C|A$ holds for the generic measure, (11) can be rewritten as (8). Moreover, this constraint implies that (12a) and (13a) are satisfied by the generic measure, (12b) and (13b) being satisfied if in addition $B \perp C|D$ holds. By symmetry, it is obvious that the set of dual constraints $A \perp D|B$ and $A \perp D|C$ also imply that (12) and (13) be satisfied.

Definition 5: Π_0 is the subset of Π_{MN} such that

$$(B \perp C|A \text{ and } B \perp C|D) \text{ or } (A \perp D|B \text{ and } A \perp D|C).$$

Then, we can put a formal definition upon the term PRF.

Definition 6: A random field \mathbf{X} on Λ is a positive PRF (PRF $^+$) iff $P(\mathbf{x})$ can be expressed as (11) where the generic measure belongs to Π_0 .

Obviously, since $\Pi_0 \subset \Pi_{MN}$, any PRF $^+$ is a stationary MRF $_2^2$.

Theorem 3 is reminiscent of [11, Corollary 3]. More precisely, it can be shown that conditions (12) and (13) are equivalent to [11, Conditions (45) and (46)]. However, Theorem 3 states that these conditions are equivalent to stationarity whereas the equivalence in [11, Corollary 3] is restricted to stationary GRF's with a homogeneous LTF. Actually, taking an algebraic path involving the Hammersley–Clifford theorem, it can be shown that every stationary GRF holds a homogeneous LTF description through canonical Gibbs potentials. Indirectly, this provides an alternate proof of Theorem 3.

Provided a generic measure is selected according to (12) and (13), (11) gives the exact probability of a given image \mathbf{x} , whereas in the classical MRF theory one handles this

$$P(\mathbf{x}) = \frac{\prod_{m=1}^{M-1} \prod_{n=1}^{N-1} [x_{mn} \ x_{m,n+1}]}{\prod_{m=2}^{M-1} \prod_{n=2}^{N-1} [x_{mn}]} \frac{\prod_{m=1}^{M-1} \prod_{n=1}^{N-1} [x_{m+1,n} \ x_{m+1,n+1}]}{\prod_{m=2}^{M-1} \prod_{n=2}^{N-1} [x_{m+1,n}]} \quad (11)$$

quantity up to a normalizing factor, i.e., the partition function. Therefore, one might expect that using stationary MRF_2^2 's will yield consistent technical simplification over general MRF 's. However, it should be emphasized that Theorem 3 characterizes stationary MRF_2^2 's without providing any constructive principle. In some sense, the complexity of explicit computation of the partition function is transferred to the resolution of implicit equations (12) and (13). The purpose of Section IV is to derive simpler forms of these constraints and to check whether they might reduce to Pickard constraints in some cases of general interest. This last question is sensible since the set of PRF^+ 's is the only acknowledged subset of the set of stationary MRF_2^2 's.

IV. CHARACTERIZATION OF THE GENERIC MEASURE

Implicit equations (12) and (13) look intricate despite their simple intuitive meaning (see Appendix C). Moreover, these constraints depend on the size (M, N) of the lattice, and they impose similar constraints for all $M' \leq M$ and $N' \leq N$ as special cases. Here, our goal is to derive constraints as independent from M and N as possible.

In the cases we will examine: namely, the symmetric, Gaussian, and binary cases, this goal is achieved since the necessary condition for (3×3) or (4×4) lattices also proves to be sufficient for all larger lattices. In the symmetric and Gaussian cases this is due to the fact that conditions on a (3×3) lattice imply Pickard conditions, which are known to yield stationary fields on \mathbb{Z}^2 . The study of the binary case is more complex and requires additional investigation of constraints on a (4×4) lattice.

To some extent, the main results presented in this section rely on constraint ((13a), $M = 3$) on the generic measure

$$\forall (a, c, e) \in E^3, \quad \sum_{bdf} \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix}}{\begin{bmatrix} c & d \end{bmatrix}} = \frac{\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} c \\ e \end{bmatrix}}{\begin{bmatrix} c \end{bmatrix}}. \quad (15)$$

However, this constraint is not very useful in the present form. A much more convenient form may be derived, as stated in the following lemma.

Lemma 3: Equation (15) is equivalent to (16) shown at the bottom of this page.

Proof: Development of the product under the summation sign in (16) followed by multiplication of both sides by $\begin{bmatrix} c \end{bmatrix}$ yields

$$\sum_d \frac{\begin{bmatrix} a \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ e \end{bmatrix}}{\begin{bmatrix} c & d \end{bmatrix}} - \frac{\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} c \\ e \end{bmatrix}}{\begin{bmatrix} c \end{bmatrix}} = 0$$

which is equivalent to (15). \square

A. Symmetric Stationary MRF_2^2 's Are PRF^+ 's

The term symmetric means invariant by reversion of the horizontal axis and by reversion of vertical axis. This is an important subcase since most GRF models used in image processing are symmetric [16]. We have the following theorem.

Theorem 4: An MRF_2^2 is symmetric and stationary on the finite rectangular lattice Λ if and only if it is a symmetric PRF^+ .

Proof: Since PRF^+ 's are stationary the sufficient condition is obvious. We now prove the necessary condition. Assume \mathbf{X} is a symmetric stationary MRF_2^2 . Then, the generic measure fulfills ((13a), $M = 3$) as a special case. According to Lemma 3 this is equivalent to (16) for all $(a, c, e) \in E^3$. Now, symmetry of \mathbf{X} implies symmetry of the generic measure and (16) implies, for $e = a$

$$\sum_d \frac{\left(\begin{bmatrix} a \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ a \end{bmatrix} \right)^2}{\begin{bmatrix} c & d \end{bmatrix}} = 0. \quad (17)$$

Therefore, for all a, c , and d , we have

$$\begin{bmatrix} a \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} \quad (18)$$

i.e., $A \perp D|C$. By symmetry, this implies all other Pickard-type conditions. Therefore, \mathbf{X} is a PRF^+ . \square

Following [11] let the term isotropic denote "invariant under the symmetries of the square" (i.e., horizontal, vertical, and diagonal axes). Then Theorem 4 implies, as a special case, that an isotropic stationary MRF_2^2 is an isotropic PRF^+ , i.e., according to the terminology introduced by Pickard [7], a "curious lattice" process.

B. A Stationary Gaussian MRF_2^2 Is a PRF^+

All previous results generalize to the case of nonsingular Gaussian MRF_2^2 's (GMRF_2^2 's). In this subsection we provide even stronger results for GMRF_2^2 's. As noted by Pickard [8], a Gaussian and stationary generic measure satisfies the following equivalences:

$$A \perp D|B \Leftrightarrow A \perp D|C \quad (19a)$$

$$B \perp C|A \Leftrightarrow B \perp C|D. \quad (19b)$$

Hence, a stationary GMRF_2^2 is a PRF^+ if at least one of the four conditions holds. In fact, we have a stronger equivalence.

Theorem 5: A GMRF_2^2 is stationary on the finite rectangular lattice Λ if and only if it is a Gaussian PRF^+ . \square

$$\forall (a, c, e) \in E^3, \quad \sum_d \frac{\left(\begin{bmatrix} a \\ c & d \end{bmatrix} - \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} \right) \left(\begin{bmatrix} c & d \\ e \end{bmatrix} - \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} c \\ e \end{bmatrix} \right)}{\begin{bmatrix} c & d \end{bmatrix}} = 0. \quad (16)$$

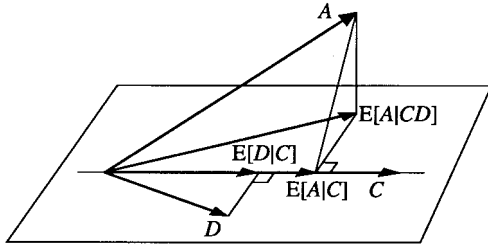


Fig. 3. Illustration of the equality $E[A|CD] - E[A|C] = \alpha(D - E[D|C])$ in terms of linear projections.

Proof: The sufficient condition is straightforward; we prove the necessary condition. Let \mathbf{X} be a stationary GMRF $_2^2$, and consider a (3×2) sublattice. Let A, C, D , and E be RV's such that

$$P(A = a, C = c, D = d, E = e) = \begin{bmatrix} a & \\ c & d \\ & e \end{bmatrix}.$$

Then ((13a), $M = 3$) holds provided summation is formally replaced by integration. By Lemma 3, (16) holds as well for all $(a, c, e) \in \mathbb{R}^3$ and it is equivalent to

$$\int_{\mathbb{R}} \left(\begin{bmatrix} a \\ c & d \end{bmatrix} - \begin{bmatrix} a \\ c \end{bmatrix} \right) \left(\begin{bmatrix} c & d \\ e \end{bmatrix} - \begin{bmatrix} c \\ e \end{bmatrix} \right) [c|d] dd = 0. \quad (20)$$

Multiplication of the latter equality by $ae[c]$ followed by integration over a, c , and e yields

$$E[(EA|CD] - E[A|C])(E[E|CD] - E[E|C]) = 0. \quad (21)$$

Due to the Gaussian assumption, an elementary linear projection argument (see Fig. 3) can be used to assess the existence and uniqueness of a couple of scalars (α, β) such that

$$\begin{cases} E[A|CD] - E[A|C] = \alpha(D - E[D|C]) \\ E[E|CD] - E[E|C] = \beta(D - E[D|C]). \end{cases}$$

Injection of the latter in (21), and accounting for

$$E[(D - E[D|C])^2] > 0$$

(nonsingular covariance) implies whether $E[A|CD] = E[A|C]$ or $E[E|CD] = E[E|C]$, i.e., whether $A \perp D|C$ or $D \perp E|C$. Due to the stationarity of the measure, $D \perp E|C$ is equivalent to $B \perp C|A$. Hence, accounting for remark (19), \mathbf{X} is a PRF $^+$. \square

As a consequence of Theorem 5, the class of stationary Gaussian first-order MRF's reduces to trivial distributions.

Corollary 2: The class of stationary Gaussian MRF $_1^2$'s on Λ reduces to collections of side-by-side independent MC's.

Proof: By Theorem 5, we only have to show that there exists no nontrivial first-order PRF $^+$. Notations are taken from [8, p. 669]. Any stationary Gaussian measure on Λ_{22} is determined by common mean and variance and the normalized covariance

$$\Gamma = \begin{pmatrix} 1 & \rho_{01} & \rho_{10} & \rho_{11} \\ \rho_{01} & 1 & \rho_{-11} & \rho_{10} \\ \rho_{10} & \rho_{-11} & 1 & \rho_{01} \\ \rho_{11} & \rho_{10} & \rho_{01} & 1 \end{pmatrix}. \quad (22)$$

Pickard condition $B \perp C|A$ reads $\rho_{-11} = \rho_{01}\rho_{10}$. In the Gaussian PRF $^+$ case, *first-order* means that entries (1, 4) and (2, 3) in Γ^{-1} should vanish. After some computations, this boils down to

$$\begin{aligned} \rho_{11} &= \rho_{01}\rho_{10}(2 - \rho_{01}^2 - \rho_{10}^2)/(1 - \rho_{01}^2 - \rho_{10}^2), \\ 0 &= \rho_{01}\rho_{10}(1 - \rho_{01})^2(1 - \rho_{10}^2)/(1 - \rho_{01}^2\rho_{10}^2)^2. \end{aligned} \quad (23)$$

Since nonsingular covariance imposes $|\rho_{01}| < 1$ and $|\rho_{10}| < 1$, (23) reduces to degenerate case $\rho_{10}\rho_{01} = 0$. \square

Before closing this subsection, let us mention that a recent paper [17] establishes a one-to-one explicit correspondence between the set of parameters describing the covariance of any PRF $^+$ and the set of valid parameters of second-order quarter-plane AR's (three linear prediction plus one variance). Since the latter set is a tetrahedron, it has a much simpler description than the equivalent set of covariance parameters.

C. Stationary Binary MRF $_2^2$'s

Both previous subsections seem to indicate that Pickard conditions are intrinsically linked to stationarity. Surprisingly, investigation of the binary case reveals the existence of a new class of nontrivial stationary binary MRF $_2^2$'s (BMRF $_2^2$'s) which is the unique alternative to binary PRF $^+$'s.

1) *Stationary BMRF $_2^2$'s on Λ_{33} :* Let us consider the case of a (3×3) lattice. As shown in Section IV-A, ((13a), $M = 3$) is equivalent to (16). First, we specialize (16) to the binary case. Then we combine similar specialized versions of ((12a), $N = 3$), ((12b), $N = 3$), and ((13b), $M = 3$) to further characterize the set of generic measures Π_{33} .

We now introduce a compact notation for the factors appearing in (16). Let $0_{\mathbf{L}}(\cdot, \cdot)$ and $1_{\mathbf{L}}(\cdot, \cdot)$ be two real functions defined on $\{0, 1\}^2$ as follows:

$$\forall (a, c, d) \in \{0, 1\}^3, \quad c_{\mathbf{L}}(a, d) \triangleq \begin{bmatrix} a \\ c & d \end{bmatrix} - [c|d] \begin{bmatrix} a \\ c \end{bmatrix}. \quad (24)$$

Using this definition, it is obvious that $c_{\mathbf{L}}(a, d) + c_{\mathbf{L}}(a, \bar{d}) = 0$, where $\bar{d} \triangleq 1 - d$. Moreover, this property generalizes as

$$\forall (a, c, d), \quad c_{\mathbf{L}}(a, d) = -c_{\mathbf{L}}(a, \bar{d}) = -c_{\mathbf{L}}(\bar{a}, d) = c_{\mathbf{L}}(\bar{a}, \bar{d}). \quad (25)$$

In particular, if $\exists(a, d)$ such that $c_{\mathbf{L}}(a, d) = 0$ the same equality holds for all a and d , i.e., $c_{\mathbf{L}} \equiv 0$. In a similar manner, we define functions $1_{\mathbf{r}}, 0_{\mathbf{r}}, 1_{\mathbf{t}}, 0_{\mathbf{t}}, 1_{\mathbf{l}},$ and $0_{\mathbf{l}}$. This allows us to simplify the expressions of several properties. For instance, $B \perp C|A$ can be written $0_{\mathbf{r}} \equiv 1_{\mathbf{r}} \equiv 0$.

We have the following equivalence for (16):

Lemma 4: A generic binary measure satisfies (16) if and only if

$$(0_{\mathbf{L}} \equiv 0 \text{ or } 0_{\mathbf{r}} \equiv 0) \text{ and } (1_{\mathbf{L}} \equiv 0 \text{ or } 1_{\mathbf{r}} \equiv 0).$$

Proof: Using compact notations, (16) reads

$$\forall (a, c, e) \in \{0, 1\}^3, \frac{c_L(a, d)c_r(d, e)}{[c\bar{d}]} + \frac{c_L(a, \bar{d})c_r(\bar{d}, e)}{[c\bar{d}]} = 0.$$

Since

$$\begin{aligned} c_L(a, d) &= -c_L(a, \bar{d}) \\ c_r(d, e) &= -c_r(\bar{d}, e) \end{aligned}$$

and

$$[c\bar{d}] + [c\bar{d}] = 1$$

(16) is equivalent to

$$c_L(a, d)c_r(d, e) = 0, \quad \forall (a, c, e) \in \{0, 1\}^3.$$

Since functions c_L and c_r identically vanish when they vanish once, Lemma 4 holds. \square

According to Lemma 4 there are four different cases, namely, $0_L \equiv 1_L \equiv 0$, $0_r \equiv 1_r \equiv 0$, $0_L \equiv 1_r \equiv 0$, or $1_L \equiv 0_r \equiv 0$, and the last two are not Pickard-like properties. However, a stationary BMRF_2^2 must additionally satisfy ((12a), $N = 3$), ((12b), $N = 3$), and ((13b), $M = 3$). One could expect that the whole combination yields Pickard constraints only. This is not so, and careful inspection shows the existence of following alternative (see Appendix E).

Theorem 6: A BMRF_2^2 is stationary on Λ_{33} if at least one of the following assertions is true:

$$1_r \equiv 0_r \equiv 1_L \equiv 0_L \equiv 0 \quad (26a)$$

$$1_L \equiv 0_L \equiv 1_r \equiv 0_r \equiv 0 \quad (26b)$$

$$1_r \equiv 1_L \equiv 0_r \equiv 0_L \equiv 0 \quad (26c)$$

$$0_r \equiv 0_L \equiv 1_r \equiv 1_L \equiv 0. \quad (26d)$$

2) *Stationary BMRF_2^2 's on Λ , M , and $N \geq 4$:* Since (26a) and (26b) correspond to PRF's, one may still wonder whether there exist generic measures that satisfy (26c) or (26d), and, be the answer positive, if (26c) or (26d) is a sufficient condition for stationarity of \mathbf{X} on lattices greater than Λ_{33} . We will see later in this section that the answer to the first question is positive, while the second issue is circumvented by a different set of necessary and sufficient conditions valid on lattices greater than Λ_{33} . These conditions are stated formally in the following lemmas.

Lemma 5: A generic binary measure satisfies ((13a), $M = 4$) if and only if one of the following assertions is true:

$$B \perp C|A \text{ or } A \perp D|C \quad (27a)$$

$$\begin{bmatrix} 1 & \bar{b} \\ 0 & \bar{d} \end{bmatrix} = [1\bar{b}][0\bar{d}] \quad (27b)$$

$$\begin{bmatrix} 0 & \bar{b} \\ 1 & \bar{d} \end{bmatrix} = [0\bar{b}][1\bar{d}] \quad (27c)$$

Moreover, we have the following sufficient condition.

Lemma 6: If a generic binary measure satisfies ((13a), $M = 4$) then it satisfies (13a) for all $M \geq 3$.

Proofs for Lemmas 5 and 6 are presented in Appendices F and G, respectively.

Using Lemma 6 and a proper reordering of the variables proves that ((12), $M = 4$) and ((13), $N = 4$) imply (12) and (13) for all M and $N \geq 4$, in other terms $\forall (M, N) \geq 4$, $\Pi_{MN} = \Pi_{44}$.

Now, we have to further characterize Π_{44} : ((13b), $M = 4$), ((12a), $N = 4$), and ((12b), $N = 4$) yield conditions that are similar to (27). Inspection of the different cases along the same lines as in Appendix E shows that there are only two classes of non-Pickard generic measures, denoted Π_+ and Π_- , which are dual in the following sense:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Pi_+ \Leftrightarrow \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix} \in \Pi_-.$$

Definition 7: Π_+ is the subset of Π_{22} defined by the following properties:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \quad (28a)$$

$$\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} = \frac{[1 \ 0][1 \ b][0 \ d]}{[0][1]}. \quad (28b)$$

It is shown in Appendix H that Π_+ is a nonempty set which admits a convenient four-parameter description. We are now able to provide a complete characterization of stationary BMRF_2^2 's.

Theorem 7: A BMRF_2^2 \mathbf{X} is stationary on the finite rectangular lattice Λ (M and $N \geq 4$) if and only if $P(\mathbf{x})$ factors as (11) and the generic measure belongs to

$$\Pi_{MN} = \Pi_{44} = \Pi_0 \cup \Pi_+ \cup \Pi_-.$$

Proof: The essential steps for the necessary condition have already been given. As regards the sufficient condition, note that the ‘‘diagonal symmetry’’ (28a) implies $[a \ b] = \begin{bmatrix} a \\ b \end{bmatrix}$. Thus (28b) is equivalent to (27c), and by Lemma 6, (13a) holds for all $M \geq 3$. Moreover, (13a) and the diagonal symmetry (28a) implies (12a), (12b), and (13b). Hence, by Theorem 3, (11) defines a stationary MRF_2^2 . \square

3) *Simulation of Stationary BMRF_2^2 's:* Appendix H provides a precise though not intuitive insight into what a non-Pickard BMRF_2^2 is. In particular, one may wonder if those fields can be simulated easily, or even if they present the same unilateral structure as PRF's.

Since a stationary BMRF_2^2 \mathbf{X} is a VMC^- , the simulation may be handled row by row. The first row is a stationary MC which is easy to sample, and any other row may be simulated conditionally to the previous one. Therefore, the basic task to be performed is the generation of $\mathbf{X}_2^- | \mathbf{X}_1^- = \mathbf{x}_1^-$. We might use the technique presented in [18], which is based upon the fact that $\mathbf{X}_2^- | \mathbf{X}_1^- = \mathbf{x}_1^-$ is an inhomogeneous MC. In a first step (inhomogeneous) transition matrices are computed and stored, then these matrices are used to simulate the chain. In the binary case considered here, most of the computation

time is consumed in the first step which requires one division, four multiplications, and two additions per site. However, the particular structure of the generic measure may be used to derive a faster algorithm, which just requires logical tests and, in addition, gives further insight into the nature of those non-Pickard fields.

In the sequel, the proofs of lemmas and theorems will be omitted since they only involve routine techniques. Only the main arguments will be provided. We may suppose, without loss of generality, that the generic measure belongs to Π_+ . Therefore, it satisfies:

$$\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ d \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}. \quad (29)$$

Simulation is much simplified by the following lemma.

Lemma 7: If the generic measure satisfies (29), then the following independence property holds:

$$\begin{aligned} & \begin{bmatrix} u_1 & \cdots & u_{n-1} & 1 & 0 & u_{n+2} & \cdots & u_N \\ v_1 & \cdots & v_{n-1} & v_n & v_{n+1} & v_{n+2} & \cdots & v_N \end{bmatrix} \\ &= \begin{bmatrix} u_1 & \cdots & u_{n-1} & 1 \\ v_1 & \cdots & v_{n-1} & v_n \end{bmatrix} \begin{bmatrix} 0 & u_{n+2} & \cdots & u_N \\ v_{n+1} & v_{n+2} & \cdots & v_N \end{bmatrix}. \quad (30) \end{aligned}$$

Therefore, sampling of the second row can be split into independent sampling of smaller parts of the row, each of which being of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ v_1 & \cdots & v_J & v_{J+1} & \cdots & v_n \end{bmatrix}. \quad (31)$$

It can be proved by induction that (31) factors as

$$\left(\prod_{j=1}^{J-1} \begin{bmatrix} 0 & 0 \\ v_j & v_{j+1} \end{bmatrix} \right) \begin{bmatrix} 0 & 1 \\ v_J & v_{J+1} \end{bmatrix} \left(\prod_{j=J+1}^{n-1} \begin{bmatrix} 1 & 1 \\ v_j & v_{j+1} \end{bmatrix} \right). \quad (32)$$

The sampling scheme suggested by (32) is to sample (V_J, V_{J+1}) according to

$$\begin{bmatrix} 0 & 1 \\ v_J & v_{J+1} \end{bmatrix}$$

then to sample two homogeneous MC's, one in a backward direction with

$$\begin{bmatrix} 0 & 0 \\ v_j & v_{j+1} \end{bmatrix}$$

as transition kernel, the other one in forward direction with

$$\begin{bmatrix} 1 & 1 \\ v_j & v_{j+1} \end{bmatrix}$$

as transition kernel. Fig. 4 provides an example of a realization of a non-Pickard stationary BMRF_2^2 generated row by row according to this scheme.

In the proposed algorithm, one can set an ordering of the row such that the probability of a RV given all ‘‘past’’ states only depends on a limited set of states. In this respect, one might find that the process is unilateral. This conclusion is wrong, because the ordering of RV's on a row depends on the states of the previous one, which are random. Therefore, one

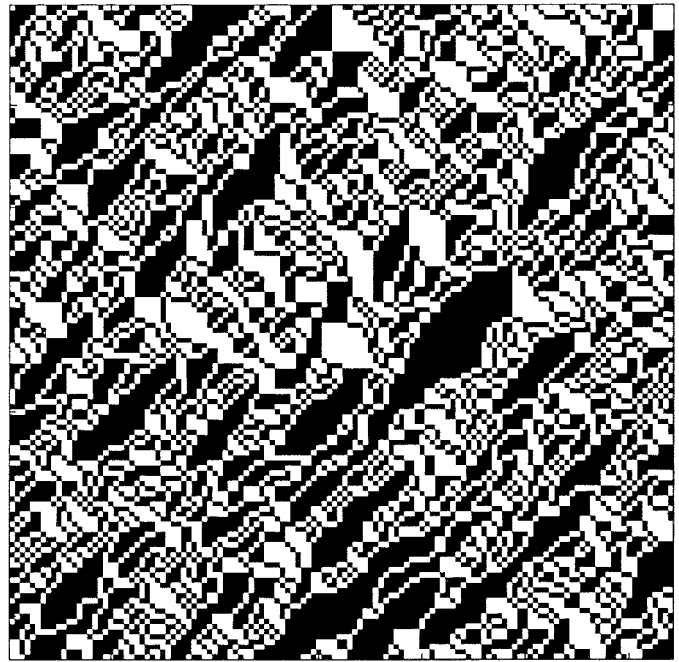


Fig. 4. Realization of a non-Pickard stationary BMRF_2^2 on a 128×128 lattice. The generic measure was selected among the class Π_+ with $p = 0.5$, $q = 0.4$, $r = s = 10^{-4}$. Note the strong anisotropic effect: patches of ones (resp., zeros) tend to spread along the main (resp., second) diagonal.

cannot set an ordering once and for all. Instead, it must be established dynamically while the samples are generated.

As such, (30)–(32) form the basis for generation of a stationary BMRF_2^2 whose number of columns is fixed in advance: it is generated row by row given the previous row. However, given the symmetries in the generic measure, column-wise counterparts of these equations do exist. Therefore, one can sample on a lattice using any previously sampled sublattice by addition of more rows or columns as suggested before. Indeed, this remark implies that a stationary BMRF_2^2 can be extended over \mathbb{Z}^2 by *compatibility* and the Kolmogorov extension theorem [19], as explained in the next section.

V. EXTENSION OF STATIONARY MRF_2^2 TO \mathbb{Z}^2

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a positive stationary measure that fulfills (12) and (13) for all M and N greater than 2, i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Pi \triangleq \bigcap_{M, N > 2} \Pi_{MN}.$$

Π is a nonempty set since $\Pi_0 \subset \Pi$. Let $(\Lambda_k)_{k>0}$ be a sequence of finite rectangular lattices growing toward \mathbb{Z}^2 . By Theorem 3, (11) enables us to construct a sequence of probability distributions P_{Λ_k} indexed on Λ_k . These distributions are compatible since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Pi$. Therefore, by the Kolmogorov extension theorem [19], the sequence P_{Λ_k} converges toward a limit distribution, denoted $P_{\mathbb{Z}^2}$. By construction, $P_{\mathbb{Z}^2}$ defines a stationary second-order positive MRF on \mathbb{Z}^2 , and all the marginals of this field on a rectangular sublattice define an MRF_2^2 . Any field on \mathbb{Z}^2 which possesses such properties will be referred to as a stationary *compatible* MRF_2^2 (CMRF_2^2). Note that the class of stationary CMRF_2^2 's is a subset of

the wider class of stationary MRF_2^2 on \mathbb{Z}^2 . Reciprocally, let $P_{\mathbb{Z}^2}$ be a stationary CMRF_2^2 on \mathbb{Z}^2 . Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote the common positive stationary measure of any 2×2 square cell. By definition, the marginal distribution P_Λ of $P_{\mathbb{Z}^2}$ on any rectangular sublattice Λ defines a stationary MRF_2^2 . Therefore, by Theorem 3, P_Λ factors as (11) in terms of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Moreover, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Pi$. In other terms there is a one-to-one correspondence between the class of stationary CMRF_2^2 's on \mathbb{Z}^2 and the set of generic measures Π .

It is clear from the work of Pickard [8] and from our results on stationary Gaussian MRF_2^2 's that a Gaussian CMRF_2^2 is a second-order quarter-plane AR process. Reciprocally, using the results of Tory and Pickard [17], any second-order quarter-plane AR is also a CMRF_2^2 . Now, consider the stationary Gaussian MRF \mathbf{X} defined on \mathbb{Z}^2 by

$$\begin{aligned} \forall (m, n) \in \mathbb{Z}^2, \forall A \subset \mathbb{Z}^2, \\ N_{mn} \subset A \Rightarrow E[X_{mn} | X_{A \setminus \{mn\}}] \\ = \alpha(X_{m-1, n} + X_{m+1, n} + X_{m, n-1} + X_{m, n+1}) \end{aligned}$$

where N_{mn} is the first-order neighborhood of site (m, n) , and α is any scalar such that $|\alpha| < 1/4$. It is well known that \mathbf{X} admits no causal representation [14]. Therefore, \mathbf{X} is not a CMRF_2^2 and the restriction of \mathbf{X} on any finite and rectangular lattice Λ cannot yield an MRF_2^2 : the first-order Markov property is lost on the boundaries of Λ .

VI. CONCLUSION

This paper addressed the characterization of second-order stationary MRF's on a finite rectangular lattice under the positivity assumption (i.e., stationary MRF_2^2 's). This problem is reminiscent of works by Pickard [8] and Goutsias [11] and it proved to be equivalent to the characterization of the second-order stationary MRF's indexed on \mathbb{Z}^2 that keep their Markov property when restricted to finite rectangular lattices. In a first step, necessary and sufficient conditions for stationarity of an MRF_2^2 were derived. These conditions underlie some very specific properties of stationary MRF_2^2 's: in particular, their restriction to any rectangular sublattice remains a stationary MRF_2^2 and Markov chains are embedded within each row or column. Moreover, stationary MRF_2^2 's are entirely described by their probability on a generic 2×2 cell subject to some consistency constraints. It is well known that a class of stationary MRF's, discovered and studied by Pickard [8] earlier, fulfills these properties. Hence, our work proves that these properties are not specific to PRF's, but of stationary MRF_2^2 's, which *a priori* define a wider set. In a second step we attempted to solve the consistency constraints for the generic measure. Due to the intricate form of this constraints no general solution was provided. However, some cases of very general interest were extensively resolved; namely, the symmetric, Gaussian, and binary stationary MRF_2^2 's. First, we proved in a straightforward manner that symmetric and Gaussian stationary MRF_2^2 's are PRF's, and are therefore unilateral. To a certain extent, these nontrivial results make the Pickard construction appear quite general. However, investigation of the binary case qualifies this statement. We were able to describe the class of non-Pickard stationary MRF_2^2 's

exhaustively and to provide a fast exact sampling algorithm for these nonunilateral fields.

These results indicate that stationarity imposes strong constraints on MRF's that reduce them to unilateral or pseudo-unilateral fields. The class of such models is relatively poor compared to general MRF's, and their practical usefulness relies on the numerical advantages offered by unilaterality: possibility of one-pass sampling, direct implementation of a full-Bayesian sampling scheme for posterior distributions (including those of MRF parameters) without having to estimate the partition function [20]. Conversely, unilateral fields are only rough models for realistic images. For instance, they are able to model large-scale clustering but strong directional effects may also be present [22]. However, when unilateral fields seem inadequate as image models, they can still be used as approximations to general MRF's, e.g., inside importance sampling schemes [23], [24].

This work leaves some issues unsolved and opens the way to further generalizations. First of all, we were not able to solve the n -ary case when $n > 2$. Although generalization of the non-Pickard binary fields to the n -ary case is straightforward, the question of exhaustivity is still open. However, the material presented here forms the basis for generalizations toward stationary MRF's indexed on finite "rectangular" subsets of \mathbb{Z}^p and/or MRF's with larger neighborhoods. For instance, it is not difficult to extend the results of Section III toward MRF_3^3 's (or even MRF_p^p 's, $p > 3$) and to fifth-order MRF's (MRF_8^8 's) indexed on a plane rectangular lattice. In particular, a stationary MRF_3^3 remains a stationary MRF_3^3 when restricted to a smaller sublattice. It could be considered as a stack of stationary MRF_2^2 's and be described in terms of a cubic $2 \times 2 \times 2$ generic measure that must fulfill some consistency constraints analog to the ones derived for stationary MRF_2^2 's.

APPENDIX A

PROOF OF COUNTEREXAMPLE 1

The potential in (10) can be expressed in matrix form as $\mathbf{x}^t Q \mathbf{x}$ where \mathbf{x} is a row-wise enumeration of the sites in Λ and Q has the following Toeplitz-bloc-Toeplitz structure:

$$Q = \begin{pmatrix} Q_1 & Q_2 & & & \\ Q_2 & Q_1 & \ddots & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & \ddots & Q_1 & Q_2 \\ & & & Q_2 & Q_1 \end{pmatrix} \quad (33)$$

$$Q_1 = \begin{pmatrix} 1 & \alpha & & \\ \alpha & 1 & \ddots & \mathbf{0} \\ & \ddots & \ddots & \ddots \\ \mathbf{0} & & \ddots & 1 & \alpha \\ & & & \alpha & 1 \end{pmatrix} \quad Q_2 = \alpha I_M. \quad (34)$$

The covariance R of \mathbf{X} is proportional to Q^{-1} . Let $A = \Lambda_{M-1, N}$ denote Λ minus the last row of Λ . The neighborhood system of the last row in \mathbf{X}_A can be inferred from the last $(N \times N)$ block in the inverse of the covariance R_A of \mathbf{X}_A . By the partitioned matrix inversion lemma, this block

is proportional to $Q_1 - \alpha^2 Q_1^{-1}$. The inverse of a tridiagonal matrix is full unless it is block-diagonal (this case is excluded here since $\alpha \neq 0$). Therefore, Q_1^{-1} is a full matrix, the whole row is connected and \mathbf{X}_A is not an MRF $_2^2$.

APPENDIX B

MARKOV-CHAIN STRUCTURE OF ANY ROW OR COLUMN

The purpose of this appendix is to prove Lemma 2, i.e., that if \mathbf{X} is a stationary MRF $_2^2$, the marginal distribution of any row and column is Markovian. First, consider the case $M = 3$. The idea underlying the proof is the following: \mathbf{X} may be considered as a VMC either by row or by column; each pair of contiguous rows or columns is also a VMC and their distributions factor in terms of the generic measure. Therefore, $P(\mathbf{x})$ admits two factorizations in terms of the generic measure, but these factorizations are equivalent descriptions of $P(\mathbf{x})$. This equivalence brings the Markovian constraint on each row and column of \mathbf{X} .

On the one hand, \mathbf{X} is a VMC $^-$, therefore,

$$P(\mathbf{x}) = P(\mathbf{x}_1^-, \mathbf{x}_2^-, \mathbf{x}_3^-) = \frac{P(\mathbf{x}_1^-, \mathbf{x}_2^-)P(\mathbf{x}_2^-, \mathbf{x}_3^-)}{P(\mathbf{x}_2^-)}. \tag{35}$$

On the other hand, $(\mathbf{X}_1^-, \mathbf{X}_2^-)$ and $(\mathbf{X}_2^-, \mathbf{X}_3^-)$ are also VMC $^+$, therefore,

$$P(\mathbf{x}) = \frac{\prod_{m=1}^2 \prod_{n=1}^{N-1} \begin{bmatrix} x_{mn} & x_{m,n+1} \\ x_{m+1,n} & x_{m+1,n+1} \end{bmatrix}}{P(\mathbf{x}_2^-) \prod_{m=1}^2 \prod_{n=2}^{N-1} \begin{bmatrix} x_{mn} \\ x_{m+1,n} \end{bmatrix}}. \tag{36}$$

In the same way, \mathbf{X} is a VMC $^+$

$$P(\mathbf{x}) = P(\mathbf{x}_1^+, \dots, \mathbf{x}_N^+) = \frac{P(\mathbf{x}_1^+, \mathbf{x}_2^+) \cdots P(\mathbf{x}_{N-1}^+, \mathbf{x}_N^+)}{P(\mathbf{x}_2^+) \cdots P(\mathbf{x}_{N-1}^+)}. \tag{37}$$

and each $(\mathbf{X}_n^+, \mathbf{X}_{n+1}^+)$ is also a VMC $^+$, so $P(\mathbf{x})$ admits an alternate expression

$$P(\mathbf{x}) = \frac{\prod_{m=1}^2 \prod_{n=1}^{N-1} \begin{bmatrix} x_{mn} & x_{m,n+1} \\ x_{m+1,n} & x_{m+1,n+1} \end{bmatrix}}{\prod_{n=2}^{N-1} \begin{bmatrix} x_{1n} \\ x_{2n} \\ x_{3n} \end{bmatrix} \prod_{n=1}^{N-1} \begin{bmatrix} x_{2n} & x_{2,n+1} \end{bmatrix}}. \tag{38}$$

Equality of the right-hand sides of (36) and (38) implies

$$\begin{aligned} [x_{21} \cdots x_{2N}] \prod_{n=2}^{N-1} \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \\ = \prod_{n=1}^{N-1} [x_{2n} \ x_{2,n+1}] \prod_{n=2}^{N-1} \begin{bmatrix} x_{1n} \\ x_{2n} \\ x_{3n} \end{bmatrix}. \end{aligned} \tag{39}$$

Summing (39) over $x_{12}, \dots, x_{1,N-1}$ yields

$$[x_{21} \cdots x_{2N}] \prod_{n=2}^{N-1} [x_{2n}] = \prod_{n=1}^{N-1} [x_{2n} \ x_{2,n+1}] \tag{40}$$

therefore,

$$P(\mathbf{x}_2^-) = [x_{21} \cdots x_{2N}] = \frac{\prod_{n=1}^{N-1} [x_{2n} \ x_{2,n+1}]}{\prod_{n=2}^{N-1} [x_{2n}]} \tag{41}$$

i.e., \mathbf{X}_2^- is an MC, and by stationarity \mathbf{X}_1^- and \mathbf{X}_3^- are also MC's.

Now, suppose \mathbf{X} is a stationary MRF $_2^2$ on an $(M \times N)$ lattice, $M, N > 2$. By Lemma 1 the restriction of \mathbf{X} to any $(3 \times N)$ sublattice is also a stationary MRF $_2^2$, so \mathbf{X}_m^- is an MC. Direct transposition from rows to columns shows that \mathbf{X}_n^+ is an MC as well.

APPENDIX C

EXPRESSION OF THE DISTRIBUTION $P(\mathbf{x})$

The purpose of this appendix is to prove Theorem 3. The necessary condition has two parts, the first consists of expression (11) for the probability distribution of \mathbf{X} in terms of the generic measure, the other part is the set of consistency constraints (12) and (13) for the generic measure.

Assume \mathbf{X} is a stationary MRF $_2^2$. By Corollary 1 \mathbf{X} is a stationary VMC $^-$

$$P(\mathbf{x}) = P(\mathbf{x}_1^-, \dots, \mathbf{x}_M^-) = \frac{P(\mathbf{x}_1^-, \mathbf{x}_2^-) \cdots P(\mathbf{x}_{M-1}^-, \mathbf{x}_M^-)}{P(\mathbf{x}_2^-) \cdots P(\mathbf{x}_{M-1}^-)}. \tag{42}$$

Let m denote any integer such that $1 \leq m \leq M - 1$. By Lemma 1 and Corollary 1, each pair of rows $(\mathbf{X}_m^-, \mathbf{X}_{m+1}^-)$ is a stationary VMC $^+$, and by Lemma 2, \mathbf{X}_m^- is a stationary MC, so that (42) yields (11).

Constraints (12) and (13) can be obtained as follows. Each pair of contiguous rows $(\mathbf{X}_m^-, \mathbf{X}_{m+1}^-)$ is a VMC $^+$

$$P(\mathbf{X}_m^- = \mathbf{u}, \mathbf{X}_{m+1}^- = \mathbf{v}) = \frac{\prod_{n=1}^{N-1} \begin{bmatrix} u_n & u_{n+1} \\ v_n & v_{n+1} \end{bmatrix}}{\prod_{n=2}^{N-1} [u_n]} \tag{43}$$

and each row follows an MC distribution, which gives, respectively, (12a) and (12b). Reasoning along the same lines for columns provides (13).

Finally, let us prove the sufficient condition: let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a generic stationary measure that satisfies (12) and (13) and let π be the measure defined on $(E^N \times E^N)$ by

$$\pi(\mathbf{u}, \mathbf{v}) = \frac{\prod_{n=1}^{N-1} \begin{bmatrix} u_n & u_{n+1} \\ v_n & v_{n+1} \end{bmatrix}}{\prod_{n=2}^{N-1} [u_n]}.$$

Then (12) implies that π is stationary and that

$$\pi(\mathbf{u}) = \frac{\prod_{n=1}^{N-1} [u_n \ u_{n+1}]}{\prod_{n=2}^{N-1} [u_n]}$$

P defined by (11) may be expressed as follows:

$$P(\mathbf{x}) = \frac{\pi(\mathbf{x}_1^-, \mathbf{x}_2^-) \cdots \pi(\mathbf{x}_{M-1}^-, \mathbf{x}_M^-)}{\pi(\mathbf{x}_2^-) \cdots \pi(\mathbf{x}_{M-1}^-)}$$

and, according to Proposition 1, \mathbf{X} forms a stationary VMC⁻. In the same way, it forms a stationary VMC⁺. Hence, from Corollary 1, \mathbf{X} is a stationary MRF₂².

APPENDIX D

PROOF OF COUNTEREXAMPLE 2

Let σ denote a measure of Π_{22} defined by (14) and let R be the covariance matrix of (A, B, C, D) . From the form of (14), we have

$$R^{-1} = \begin{pmatrix} 1 & \alpha & \alpha & 0 \\ \alpha & 1 & 0 & \alpha \\ \alpha & 0 & 1 & \alpha \\ 0 & \alpha & \alpha & 1 \end{pmatrix}$$

which allows us to express R in closed form as

$$R = \begin{pmatrix} \beta & \gamma & \gamma & \delta \\ \gamma & \beta & \delta & \gamma \\ \gamma & \delta & \beta & \gamma \\ \delta & \gamma & \gamma & \beta \end{pmatrix}$$

with

$$\beta = \frac{1 - 2\alpha^2}{1 - 4\alpha^2} \quad \delta = \frac{2\alpha^2}{1 - 4\alpha^2} \quad \gamma = \frac{-\alpha}{1 - 4\alpha^2}. \quad (44)$$

In particular, the potential of the marginal $\sigma(a, c)$ is a quadratic form

$$(a \ c) \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \quad (45)$$

where

$$\begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix} = \begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix}^{-1} = \frac{1}{\beta^2 - \gamma^2} \begin{pmatrix} \beta & -\gamma \\ -\gamma & \beta \end{pmatrix}. \quad (46)$$

Then σ generates a stationary VMC⁺ \mathbf{X} on Λ_{2N} , whose distribution is

$$P(\mathbf{x}) = \frac{\prod_{n=1}^{N-1} \sigma \begin{pmatrix} x_{1n} & x_{1n+1} \\ x_{2n} & v_{2n+1} \end{pmatrix}}{\prod_{n=2}^{N-1} \sigma \begin{pmatrix} x_{1n} \\ x_{2n} \end{pmatrix}}. \quad (47)$$

$P(\mathbf{x})$ is characterized by a quadratic form defined by the matrix Q , which is the inverse of the covariance of \mathbf{X} . From (14), (45), and (47), Q takes the form

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2 & Q_1 \end{pmatrix} = \begin{pmatrix} R_1 & R_2 \\ R_2 & R_1 \end{pmatrix}^{-1}$$

with

$$Q_1 = \begin{pmatrix} 1 & \alpha & & & \\ \alpha & 2 - \lambda & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & & 2 - \lambda & \alpha \\ & & & \alpha & 1 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} \alpha & & & & \\ & 2\alpha - \mu & & & \\ & & \ddots & & 0 \\ & 0 & & 2\alpha - \mu & \\ & & & & \alpha \end{pmatrix}.$$

σ belongs to Π_{2N} if $P(\mathbf{x}_1^-)$ defines an MC, i.e., if $R_1^{-1} = Q_1 - Q_2 Q_1^{-1} Q_2$ is tridiagonal. Now, the inverse of a tridiagonal matrix is full (except for block-diagonal cases). Therefore, Q_1^{-1} is full, and R_1^{-1} is not tridiagonal. Hence σ does not belong to Π_{2N} . In fact, there exists no stationary Gaussian MRF₁² on Λ , as a consequence of Theorem 5.

APPENDIX E

STATIONARY BMRF₂²'S ON Λ_{33}

The purpose of this Appendix is to prove Theorem 6. Suppose \mathbf{X} is a stationary BMRF₂² on Λ_{33} . By Theorem 3, the generic measure satisfies ((12a), $N = 3$), ((12b), $N = 3$), ((13a), $M = 3$), and ((13b), $M = 3$). Using the notations introduced in Section IV-C we have

$$\begin{aligned} ((12a), N = 3) &\Leftrightarrow (0_{\uparrow} \equiv 0 \text{ or } 0_{\uparrow} \equiv 0) \text{ and} \\ &\quad (1_{\uparrow} \equiv 0 \text{ or } 1_{\uparrow} \equiv 0) \end{aligned} \quad (48a)$$

$$\begin{aligned} ((12b), N = 3) &\Leftrightarrow (0_{\downarrow} \equiv 0 \text{ or } 0_{\downarrow} \equiv 0) \text{ and} \\ &\quad (1_{\downarrow} \equiv 0 \text{ or } 1_{\downarrow} \equiv 0) \end{aligned} \quad (48b)$$

$$\begin{aligned} ((13a), M = 3) &\Leftrightarrow (0_{\uparrow} \equiv 0 \text{ or } 0_{\downarrow} \equiv 0) \text{ and} \\ &\quad (1_{\uparrow} \equiv 0 \text{ or } 1_{\downarrow} \equiv 0) \end{aligned} \quad (48c)$$

$$\begin{aligned} ((13b), M = 3) &\Leftrightarrow (0_{\downarrow} \equiv 0 \text{ or } 0_{\uparrow} \equiv 0) \text{ and} \\ &\quad (1_{\downarrow} \equiv 0 \text{ or } 1_{\uparrow} \equiv 0) \end{aligned} \quad (48d)$$

where (48c) is shown in Lemma 4 whereas (48a), (48b), and (48d) result from (48c) and a proper reordering of variables in the generic measure.

Inspection of these simultaneous constraints is much simplified by the following result.

Lemma 8:

$$\begin{cases} 1_{\uparrow} \equiv 1_{\downarrow} \Leftrightarrow 0_{\uparrow} \equiv 0_{\downarrow} \\ 1_{\downarrow} \equiv 1_{\uparrow} \Leftrightarrow 0_{\downarrow} \equiv 0_{\uparrow}. \end{cases}$$

Proof: By Bayes rule

$$\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 1b \\ c \end{bmatrix} [1] + \begin{bmatrix} 0b \\ c \end{bmatrix} [0] \quad (49a)$$

$$= \begin{bmatrix} b \\ c1 \end{bmatrix} [1] + \begin{bmatrix} b \\ c0 \end{bmatrix} [0]. \quad (49b)$$

Then (49) yields

$$[0] \left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} - \begin{bmatrix} b & 0 \\ c & 1 \end{bmatrix} \right) = -[1] \left(\begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix} - \begin{bmatrix} b & 1 \\ c & 1 \end{bmatrix} \right). \quad (50)$$

On the one hand, using definition (24) of 1_r and 1_\perp , one easily obtains

$$1_r(1, 1) - 1_\perp(1, 1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

On the other hand, accounting for symmetry of $[a \ b]$ and $\begin{bmatrix} a \\ c \end{bmatrix}$

$$0_r(1, 1) - 0_\perp(1, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, by (50)

$$[0](0_r(1, 1) - 0_\perp(1, 1)) = -[1](1_r(1, 1) - 1_\perp(1, 1)).$$

Thanks to (25), the latter equality implies

$$[0](0_r - 0_\perp) \equiv -[1](1_r - 1_\perp)$$

which proves the first equivalence in Lemma 8. Proper re-ordering of the variables in the generic measure provides the second equality. \square

We now seek a generic measure that satisfies (48). Let us study the following four subcases:

$$0_r \equiv 1_r \equiv 0 \quad (51a)$$

$$0_r \equiv 0 \text{ and } 1_r \not\equiv 0 \quad (51b)$$

$$0_r \not\equiv 0 \text{ and } 1_r \equiv 0 \quad (51c)$$

$$0_r \not\equiv 0 \text{ and } 1_r \not\equiv 0. \quad (51d)$$

The first part of Theorem 6 is proved if the following assertions hold:

$$(48) \text{ and } (51a) \Rightarrow (26a) \text{ or } (26b)$$

$$(48) \text{ and } (51b) \Rightarrow (26d) \text{ or } (26b)$$

$$(48) \text{ and } (51c) \Rightarrow (26c) \text{ or } (26b)$$

$$(48) \text{ and } (51d) \Rightarrow (26b).$$

- Case (51d) is the easiest one to deal with: using (48a) and (48c) we have $1_\perp \equiv 0_\perp \equiv 1_r \equiv 0_r \equiv 0$ (26b), which yields a Pickard measure.
- Case (51a) can be decomposed into three subcases: $0_\perp \equiv 1_\perp \equiv 0$ or $0_\perp \not\equiv 0$ or $1_\perp \not\equiv 0$. By Lemma 8, this alternative is equivalent to: $0_\perp \equiv 1_\perp \equiv 0$ or ($0_\perp \not\equiv 0$ and $1_\perp \not\equiv 0$). $0_\perp \equiv 1_\perp \equiv 0$ and (51a) together form the Pickard condition (26a). The other case, $0_\perp \not\equiv 0$ and $1_\perp \not\equiv 0$ combined with (48b) and (48d) yields $1_\perp \equiv 0_\perp \equiv 1_r \equiv 0_r \equiv 0$ (26b), which is a Pickard condition too.
- In regard to case (51b), $1_r \not\equiv 0$, (48a) and (48c) imply $1_\perp \equiv 1_r \equiv 0$. Hence, by Lemma 8, $0_\perp \equiv 0_r$. Thus we may consider the alternative: $0_\perp \equiv 0_r \equiv 0$ or $0_\perp \equiv 0_r \not\equiv 0$. The first case yields the Pickard condition (26b), whereas the second case combined with (48b) and (48d) yields $0_r \equiv 0_\perp \equiv 0$. To summarize, for a non-Pickard measure we have: $1_\perp \equiv 1_r \equiv 0_r \equiv 0_\perp \equiv 0$ which

is (26d), with the additional conditions $0_\perp \equiv 0_r \not\equiv 0$ and $1_r \equiv 1_\perp \not\equiv 0$.

- Case (51c) can be deduced from (51b) by swapping the two binary states.

Note that case (51a) corresponds to the rederivation of Pickard's Theorem [8, Theorem 6]: "Provided $B \perp C|A$, a stationary BMRF $_2^2$ is necessarily a PRF $^+$."

APPENDIX F

NECESSARY CONDITIONS FOR STATIONARITY OF A BMRF $_2^2$ ON Λ_{44}

Any stationary BMRF $_2^2$ on Λ_{44} must satisfy ((13a), $M = 4$), i.e.,

$$\forall (a, c, e, g), \sum_{bfh} \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}}{\begin{bmatrix} c & d \\ e & f \end{bmatrix}} = \frac{\begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} c \\ e \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix}}{\begin{bmatrix} c \\ e \end{bmatrix}}. \quad (52)$$

The purpose of this appendix is to prove Lemma 5 which characterizes the generic measures that satisfy (52). Lemma 5 relies on the following Lemma 9 which is an extension of Lemma 3. The proof of Lemma 9 is omitted since it uses the same technique as Lemma 3.

Lemma 9: Equation (52) is equivalent to (48c) and

$$\forall (a, c, e, g), \sum_{df} \begin{bmatrix} a & \\ c & d \end{bmatrix} \left(\begin{bmatrix} c & d \\ e & f \end{bmatrix} - \begin{bmatrix} c & \\ e & f \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} \right) \begin{bmatrix} e & f \\ g & \end{bmatrix} = 0. \quad (53)$$

By analogy with $c_\perp(a, d)$, let $ce_\perp(d, f)$ denote the middle factor inside the summation (53). Clearly, in the binary case

$$ce_\perp(d, f) = -ce_\perp(\bar{d}, f) = -ce_\perp(d, \bar{f}) = ce_\perp(\bar{d}, \bar{f}). \quad (54)$$

Hence, for all (a, c, d, e, f, g) , (53) is equivalent to

$$\left(\begin{bmatrix} a & \\ c & d \end{bmatrix} - \begin{bmatrix} a & \\ c & \bar{d} \end{bmatrix} \right) ce_\perp(d, f) \left(\begin{bmatrix} e & f \\ g & \end{bmatrix} - \begin{bmatrix} e & \bar{f} \\ g & \end{bmatrix} \right) = 0. \quad (55)$$

Moreover, it can be checked that

$$c_\perp(a, d) = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} a & \\ c & d \end{bmatrix} - \begin{bmatrix} a & \\ c & \bar{d} \end{bmatrix}$$

and

$$e_r(f, g) = \begin{bmatrix} e & f \\ g & \end{bmatrix} \begin{bmatrix} e & f \\ g & \end{bmatrix} - \begin{bmatrix} e & \bar{f} \\ g & \end{bmatrix}.$$

Therefore, for all (a, c, d, e, f, g) , (55) is equivalent to

$$c_\perp(a, d) e_r(f, g) ce_\perp(d, f) = 0. \quad (56)$$

Since $0_r(a, d) 0_\perp(f, g) = 0$ and $1_r(a, d) 1_\perp(f, g) = 0$ already hold from (48c), (56) brings new constraints only when $c \neq e$. Hence, the previous results can be summarized as follows:

$$(52) \Leftrightarrow \begin{cases} (0_r \equiv 0 \text{ or } 0_\perp \equiv 0) \text{ and } (1_r \equiv 0 \text{ or } 1_\perp \equiv 0) \\ 0_\perp \equiv 0 \text{ or } 1_r \equiv 0 \text{ or } 01_\perp \equiv 0 \\ 1_\perp \equiv 0 \text{ or } 0_r \equiv 0 \text{ or } 10_\perp \equiv 0. \end{cases}$$

Inspection of the latter set of equalities yields

$$(52) \Leftrightarrow \begin{cases} 1_r \equiv 0_r \equiv 0 \text{ or } 1_L \equiv 0_L \equiv 0 \\ \text{or } 0_r \equiv 1_L \equiv 01_C \equiv 0 \\ \text{or } 1_r \equiv 0_L \equiv 10_C \equiv 0. \end{cases} \quad (57)$$

Now, it can be easily checked that the last two assertions can be rewritten, respectively, as

$$1_r \equiv 0_L \equiv 10_C \equiv 0 \Leftrightarrow \begin{bmatrix} 1 & \overline{b} \\ 0 & \overline{d} \end{bmatrix} = [1\overline{b}][0\overline{d}] \quad (58a)$$

$$0_r \equiv 1_L \equiv 01_C \equiv 0 \Leftrightarrow \begin{bmatrix} 0 & \overline{b} \\ 1 & \overline{d} \end{bmatrix} = [0\overline{b}][1\overline{d}]. \quad (58b)$$

The combination of (57) and (58) is equivalent to Lemma 5.

APPENDIX G

SUFFICIENT CONDITIONS FOR (13a)

The purpose of this appendix is to prove Lemma 6, i.e., that, in the binary case, if a generic measure satisfies ((13a), $M = 4$) then (13a) holds for all $M \geq 3$. By Lemma 5 we have

$$((13a), M = 4) \Rightarrow (27a) \text{ or } (27b) \text{ or } (27c).$$

As mentioned in Section III, $B \perp C|A$ and $A \perp D|C$ both imply (13a) for all $M \geq 3$. Therefore, we have to prove that i) (27b) implies (13a) for all $M \geq 3$ and ii) (27c) implies (13a) for all $M \geq 3$.

First, we prove i). Given a generic measure that satisfies (27b)

$$\begin{bmatrix} 1 & \overline{b} \\ 0 & \overline{d} \end{bmatrix} = [1\overline{b}][0\overline{d}]$$

let π_M define the measure on $(E^M \times E^M)$ by

$$\pi_M(\mathbf{u}, \mathbf{v}) = \frac{\prod_{m=1}^{M-1} \begin{bmatrix} u_m & v_m \\ u_{m+1} & v_{m+1} \end{bmatrix}}{\prod_{m=2}^{M-1} [u_m \ v_m]}.$$

We must prove that $\pi_M(\mathbf{u})$ factors as an MC, for all $M \geq 3$. In the sequel, we adopt the notation " u_m^x " for $u_m = x$.

The proof is obtained by induction. For $M \geq 2$ let \mathcal{P}_M and \mathcal{Q}_M , respectively, denote the following two propositions:

$$\pi_M(u_M | u_{M-1}, \dots, u_1) = \begin{bmatrix} u_{M-1} \\ \overline{u}_M \end{bmatrix} \quad (59a)$$

$$\pi_M(v_M | u_M^0, u_{M-1}, \dots, u_1) = [0\overline{v}_M]. \quad (59b)$$

Assume $(\mathcal{P}_M, \mathcal{Q}_M)$ holds for some $M \geq 2$, then by Bayes rule

$$\begin{aligned} \pi_{M+1}(u_{M+1} | u_M, \dots, u_1) \\ = \sum_{v_M} \begin{bmatrix} u_M & v_M \\ \overline{u}_{M+1} \end{bmatrix} \pi_M(v_M | u_M, u_{M-1}, \dots, u_1). \end{aligned}$$

First, consider the case $u_M = 0$. According to (59b), the last factor under the sum is simply $[0\overline{u}_M]$, hence

$$\begin{aligned} \pi_{M+1}(u_{M+1} | u_M^0, u_{M-1}, \dots, u_1) &= \sum_{v_M} \begin{bmatrix} 0 & \overline{v}_M \\ \overline{u}_{M+1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \overline{u}_{M+1} \end{bmatrix}. \end{aligned}$$

Regarding the case $u_M = 1$, recall that (27b) implies $1_r \equiv 0$ ("integrate d out of (27b)"), i.e.,

$$\begin{bmatrix} u_M & v_M \\ \overline{u}_{M+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \overline{u}_{M+1} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \pi_{M+1}(u_{M+1} | u_M^1, \dots, u_1) \\ = \begin{bmatrix} 1 \\ \overline{u}_{M+1} \end{bmatrix} \sum_{v_M} \pi_M(v_M | u_M^1, u_{M-1}, \dots, u_1) \\ = \begin{bmatrix} 1 \\ \overline{u}_{M+1} \end{bmatrix}. \end{aligned}$$

Hence \mathcal{P}_{M+1} holds. Now, by Bayes rule

$$\begin{aligned} \pi_{M+1}(v_{M+1} | u_{M+1}^0, u_M, \dots, u_1) \\ = \frac{\pi_{M+1}(v_{M+1}, u_{M+1}^0 | u_M, \dots, u_1)}{\pi_{M+1}(u_{M+1}^0 | u_M, \dots, u_1)}. \end{aligned}$$

Since \mathcal{P}_{M+1} holds, the denominator is simply

$$\pi_{M+1}(u_{M+1}^0 | u_M).$$

By Bayes rule, the numerator factors as follows:

$$\begin{aligned} \pi_{M+1}(v_{M+1}, u_{M+1}^0 | u_M, \dots, u_1) \\ = \sum_{v_M} \begin{bmatrix} u_M & v_M \\ 0 & \overline{v}_{M+1} \end{bmatrix} \pi_M(v_M | u_M, \dots, u_1). \end{aligned}$$

Since \mathcal{Q}_M holds, when $u_M = 0$ the last factor under the sum is simply $[0\overline{u}_M]$. Hence

$$\begin{aligned} \pi_{M+1}(v_{M+1}, u_{M+1}^0 | u_M^0, \dots, u_1) &= \sum_{v_M} \begin{bmatrix} 0 & \overline{v}_M \\ 0 & \overline{v}_{M+1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0\overline{v}_{M+1} \end{bmatrix}. \end{aligned}$$

Henceforth

$$\begin{aligned} \pi_{M+1}(v_{M+1} | u_{M+1}^0, u_M^0, \dots, u_1) &= \frac{\begin{bmatrix} 0 \\ 0\overline{v}_{M+1} \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \\ &= \begin{bmatrix} 0 \\ 0\overline{v}_{M+1} \end{bmatrix}. \end{aligned}$$

Since (27b) implies $0_L \equiv 0$ ("integrate b out of (27b)"), we finally have

$$\begin{aligned} \pi_{M+1}(v_{M+1} | u_{M+1}^0, u_M^0, u_{M-1}, \dots, u_1) &= \begin{bmatrix} 0 \\ 0\overline{v}_{M+1} \end{bmatrix} \\ &= [0\overline{v}_{M+1}]. \end{aligned}$$

Hence \mathcal{Q}_{M+1} holds when $u_M = 0$. Note that we did not use (27b) fully, but only the weaker properties $1_r \equiv 0_L \equiv 0$. One can easily check that (27c) is equivalent to

$$\begin{bmatrix} 1 & b \\ 0 & \overline{d} \end{bmatrix} = [0\overline{d}][1\overline{0}].$$

Therefore,

$$\begin{aligned} \pi_{M+1}(v_{M+1}, u_{M+1}^0 | u_M^1, u_{M-1}, \dots, u_1) \\ = [0\overline{v}_{M+1}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sum_{v_M} \pi_M(v_M | u_M^1, u_{M-1}, \dots, u_1) \\ = [0\overline{v}_{M+1}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

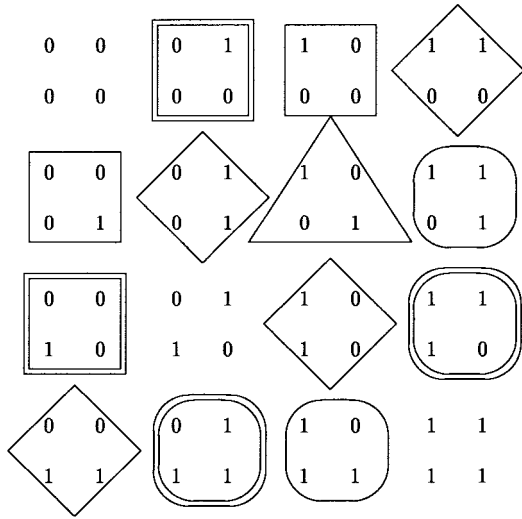


Fig. 5. Items surrounded by the same icon are diagonal-symmetric and are given the same probability. Items surrounded by a single line are obtained through (28b) and symmetries (28a), and their probability is controlled by p and q . The other probabilities are given through two additional parameters r and s . Items surrounded by a double line are obtained in a straightforward manner from r and s , whereas the three remaining items are derived using normalization constraints.

which in turn yields

$$\pi_{M+1}(v_{M+1}|u_{M+1}^0, u_M^1, u_{M-1}, \dots, u_1) = [0\overline{v_{M+1}}].$$

Hence \mathcal{Q}_{M+1} holds.

We have proved $(\mathcal{P}_M, \mathcal{Q}_M) \Rightarrow (\mathcal{P}_{M+1}, \mathcal{Q}_{M+1})$. Since $(\mathcal{P}_2, \mathcal{Q}_2)$ holds, $(\mathcal{P}_M, \mathcal{Q}_M)$ holds for all $M \geq 2$. It follows from (59a) that $\pi_M(\mathbf{u})$ factors as an MC, and, therefore, that (13a) holds for all $M \geq 3$.

By simple complementation we also obtain ii): (27c) implies (13a) for all $M \geq 3$.

APPENDIX H DESCRIPTION OF Π_+

Let us recall that Π_+ is the set of stationary measures on Λ_{22} defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \quad (28a)$$

$$\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} = \frac{[1 \ 0][1 \ b][0 \ d]}{[0][1]}. \quad (28b)$$

We will see in this appendix that Π_+ admits a fairly simple parameterization.

A measure on $\{0, 1\}^4$ is defined by sixteen positive numbers whose sum equals one. As sketched in Fig. 5, nine of these numbers are defined by $[a \ b]$ through (28). Therefore, it seems natural to parameterize this measure first. Since it is stationary, it is entirely defined by, say, the probabilities of a change of state

$$\begin{cases} [1\overline{0}] = p, & 0 < p < 1 \\ [0\overline{1}] = q, & 0 < q < 1. \end{cases} \quad (60)$$

Now, seven unspecified probabilities remain. Let r and s be defined as

$$\begin{cases} r = \begin{bmatrix} 0 & 0 \\ \overline{1} & 0 \end{bmatrix}, & 0 < r < 1 \\ s = \begin{bmatrix} 1 & 1 \\ 1 & \overline{0} \end{bmatrix}, & 0 < s < 1. \end{cases}$$

Clearly, these parameters enable six more probabilities to be set

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = r \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = r \frac{[0 \ 0]^2}{[0]} = rp(1-q)^2/(p+q) \quad (61a)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \overline{0} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (1-r) \frac{[0 \ 0]^2}{[0]} = (1-r)p(1-q)^2/(p+q) \quad (61b)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = s \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = s \frac{[1 \ 1]^2}{[1]} = sq(1-p)^2/(p+q) \quad (61c)$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \overline{1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = (1-s) \frac{[1 \ 1]^2}{[1]} = (1-s)q(1-p)^2/(p+q). \quad (61d)$$

Finally, summation over the probabilities indexed by the second column of the block matrix of Fig. 5 yields

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = [0 \ 1]. \quad (62)$$

Using (28), (61a), and (61c) in addition to (62) yields

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [0 \ 1](1 - [1\overline{1}][0\overline{0}]) - r \frac{[0 \ 0]^2}{[0]} - s \frac{[1 \ 1]^2}{[1]} \quad (63)$$

and r and s should additionally satisfy

$$\frac{[0 \ 0]^2}{[0]} r + \frac{[1 \ 1]^2}{[1]} s < [0 \ 1](1 - [1\overline{1}][0\overline{0}]).$$

This expression may be rewritten in terms of p, q, r , and s

$$p(1-q)^2 r + q(1-p)^2 s < pq(1 - (1-p)(1-q)). \quad (64)$$

Reciprocally, let $(p, q, r, s) \in (0, 1)^4$ so that constraint (64) be satisfied.⁴ Then (28), (60), (61), and (63) define a stationary generic measure.

Before ending this appendix, we have to point out that Π_+ still contains some Pickard measures, which correspond to $s = pq/(1-p)$ or $r = pq/(1-q)$. Therefore, in order to get actual non-Pickard measures, one should check that none of these equalities holds.

⁴The constraint is sufficiently simple to check that such quantities exist in $(0, 1)^4$.

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