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Convergence of Conjugate Gradient Methods with a Closed-Form Stepsize Formula

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Abstract Conjugate gradient methods are efficient methods for minimizing differentiable objective functions in large dimension spaces. However, converging line search strategies are usually not easy to choose, nor to implement. Sun and colleagues (Ann. Oper. Res. 103:161–173, 2001; J. Comput. Appl. Math. 146:37–45, 2002) introduced a simple stepsize formula. However, the associated convergence domain happens to be overrestrictive, since it precludes the optimal stepsize in the convex quadratic case. Here, we identify this stepsize formula with one iteration of the Weiszfeld algorithm in the scalar case. More generally, we propose to make use of a finite number of iterates of such an algorithm to compute the stepsize. In this framework, we establish a new convergence domain, that incorporates the optimal stepsize in the convex quadratic case.

Keywords Conjugate gradient methods · Convergence · Stepsize formulas · Weiszfeld method

1 Introduction

Let us consider the following unconstrained minimization problem: $\min_{x\in\mathbb{R}^n} \mathcal{J}(x)$, where $\mathcal{J}$ is a differentiable objective function. In the implementation of a conjugate gradient (CG) method, the stepsize strategy often incorporates a stopping criterion such as to satisfy the Wolfe conditions [3]. For instance, the Armijo condition is used

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as stopping criterion in [4]. Most recently, a simple stepsize formula was proposed by Sun and Zhang [1] and by Chen and Sun [2] for several CG methods. Its distinctive feature it to yield convergence results without any stopping condition. Here, we pursue the same direction, by proposing a generalized stepsize formula. We also reexamine the convergence conditions, which leads us to a broadened convergence domain for several types of conjugacy.

In this paper, we restrict ourselves to the following family of CG algorithms:

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1}$$

$$c_k = -g_k + \beta_k d_{k-1}, \tag{2}$$

$$d_k = -c_k \,\mathrm{sign}(g_k^{\mathrm{t}} c_k), \tag{3}$$

where the superscript t stands for transpose, $k \in \mathbb{N}$, $g_k = \nabla \mathcal{J}(x_k)$ and with the conjugacy formulas

$$\beta_0 = 0, \qquad \beta_k = \beta_k^{\mu_k,\omega_k} = g_k^{\mathrm{t}} y_{k-1}/D_k, \quad \forall k > 0, \tag{4}$$

$$D_k = (1-\mu_k-\omega_k)\|g_{k-1}\|^2 + \mu_k d_{k-1}^{\mathrm{t}} y_{k-1} - \omega_k d_{k-1}^{\mathrm{t}} g_{k-1}, \tag{5}$$

where $\|\cdot\|$ is the Euclidean norm, $y_{k-1} = g_k - g_{k-1}$, $\mu_k \in [0,1]$, and $\omega_k \in [0, 1-\mu_k]$. Let us remark that the search direction d_k is defined such that $g_k^{\mathrm{t}} d_k \le 0$.

Expression (4) is taken from [2]. It induces a subset of a larger family of nonlinear CG methods, as defined by Dai and Yuan in [5]. The following versions are covered:

$$\beta_k^{1,0} = \beta_k^{\mathrm{HS}} = g_k^{\mathrm{t}} y_{k-1}/d_{k-1}^{\mathrm{t}} y_{k-1}, \qquad \text{Hestenes-Stiefel [6]},$$

$$\beta_k^{0,0} = \beta_k^{\mathrm{PRP}} = g_k^{\mathrm{t}} y_{k-1}/\|g_{k-1}\|^2, \qquad \text{Polak-Ribière-Polyak [7, 8]},$$

$$\beta_k^{0,1} = \beta_k^{\mathrm{LS}} = -g_k^{\mathrm{t}} y_{k-1}/d_{k-1}^{\mathrm{t}} g_{k-1}, \quad \text{Liu-Storey [9]}.$$

Other important cases fall outside the present study, such as the Fletcher-Reeves method [10].

On the other hand, we focus on the following stepsize strategy:

$$\alpha_k = \alpha_k^1 = 0, \quad \text{if } d_k = 0; \tag{6}$$

otherwise,

$$\alpha_k^0 = 0, \tag{7a}$$

$$\alpha_k^{i+1} = \alpha_k^i - \theta d_k^{\mathrm{t}} \nabla \mathcal{J}(x_k + \alpha_k^i d_k)/d_k^{\mathrm{t}} Q_k^i d_k, \quad i \in \{0,\ldots,I-1\}, \tag{7b}$$

$$\alpha_k = \alpha_k^I, \tag{7c}$$

where $I \in \mathbb{N} - \{0\}$, $\theta \in \mathbb{R}$ is a parameter, $(Q_k^i) \in \mathbb{R}^{n\times n}$ is a series of symmetric positive-definite (SPD) matrices with a uniformly bounded spectrum and a strictly positive lower bound, i.e., there exist $\nu_1, \nu_2 \in \mathbb{R}$ with $\nu_2 \ge \nu_1 > 0$ such that

$$\nu_1 \|v\|^2 \le v^{\mathrm{t}} Q_k^i v \le \nu_2 \|v\|^2, \quad \forall k \in \mathbb{N}, \forall i \in \{0,\ldots,I-1\}, \forall v \in \mathbb{R}^n. \tag{8}$$

The fixed number of iterations of (7b) yields a family of CG methods with a *closed-form stepsize formula* (CFSF). In the case $I = 1$, (7b) boils down to

$$\alpha_k = \alpha_k^1 = -\theta g_k^{\mathrm{t}} d_k / d_k^{\mathrm{t}} Q_k^0 d_k, \tag{9}$$

which is exactly the formula introduced in [1, 2]. According to (3), note that the latter expression for α_k is nonnegative provided that $\theta \geq 0$.

To ensure convergence, the condition $\theta \in (0, \nu_1/\mu)$ is introduced in [1, 2], where μ is a Lipschitz constant. In Sect. 5, this condition is shown to be overrestrictive, so that (9) yields too small steps. This is obvious in the convex quadratic case, since the optimal stepsize $\theta = 1$ does not belong to the interval $(0, \nu_1/\mu) \subset (0, 1)$.

In this paper, we propose relaxed convergence conditions. In particular, the optimal stepsize becomes admissible in the convex quadratic case. The key ingredient we incorporate consists in approximating $\mathcal{J}$ by a convex quadratic function from above, which is the basic principle of the Weiszfeld method [11, 12]. First of all, we put forward that the stepsize formula proposed in [1, 2] identifies with one iteration of Weiszfeld algorithm in the scalar case. More generally, our iterated version (7b) corresponds to a fixed number of the same scalar algorithm. The majorizing convex quadratic approximation framework provides altered convergence conditions compared to the conditions found in [1, 2]: in particular, $\theta \in (0, \nu_1/\mu)$ is replaced by $\theta \in (0, 2)$ for any finite value of I.

The paper is organized as follows. Some preliminary results on the family of CG methods with the CFSF (6, 7) are given in Sect. 2. We also introduce the additional assumption of a majorizing convex quadratic function that allows us to make the connection between the CFSF and the scalar Weiszfeld method. Section 3 gathers some properties concerning the stepsize series generated by (7) useful for the next section. Section 4 includes the main convergence properties of the two-parameter family of CG methods defined by (1–7). Finally, discussions on the convex quadratic case, the general case and the case of edge preserving image restoration are given in Sect. 5.

2 Preliminaries

Let N be a neighborhood of the level set $L = \{x \in \mathbb{R}^n | \mathcal{J}(x) \leq \mathcal{J}(x_0)\}$, which is assumed bounded in the sequel. The following assumption is also adopted.

Assumption A1 $\mathcal{J} : \mathbb{R}^n \mapsto \mathbb{R}$ is differentiable on N and $\nabla\mathcal{J}$ is Lipschitz continuous on N with the Lipschitz constant $\mu > 0$,

$$\|\nabla\mathcal{J}(x) - \nabla\mathcal{J}(x')\| \leq \mu\|x - x'\|, \quad \forall x, x' \in N.$$

In short, $\mathcal{J}$ is μ-$\mathcal{LC}^1$.

In the sequel, Assumption A1 will appear to be sufficient for the global convergence of the CG method when $\mu_k = 0$ and $\omega_k \in [0, 1]$, which encompasses the PRP and the LS cases, but not the HS case. Let us consider the following stronger assumption to deal with the more general case $\mu_k \in [0, 1]$, $\omega_k \in [0, 1 - \mu_k]$.

Assumption A2 Assumption A1 holds and $\mathcal{J}$ is strongly convex on N: there exists $\lambda > 0$ such that

$$[\nabla\mathcal{J}(x) - \nabla\mathcal{J}(x')]^{\mathrm{t}}(x - x') \geq \lambda \, \|x - x'\|^2, \quad \forall x, x' \in N.$$

Note that Assumption A2 implies that L is bounded since a strongly convex function has bounded level sets. Finally, let us introduce convex quadratic majorizing functions through the following assumption.

Assumption A3 There exists a series of SPD matrices (Q_k^i) such that

$$\widehat{\mathcal{J}}_k^i(x', x) \geq \mathcal{J}(x'), \quad \forall x, x' \in N, \tag{10}$$

for all $k \in \mathbb{N}$, $i \in \{0, \dots, I-1\}$, where

$$\widehat{\mathcal{J}}_k^i(x', x) = \mathcal{J}(x) + (x' - x)^{\mathrm{t}}\, \nabla\mathcal{J}(x) + (x' - x)^{\mathrm{t}}\, Q_k^i\, (x' - x)/2. \tag{11}$$

For sake of notational simplicity, let $f(\alpha) = \mathcal{J}(x_k + \alpha d_k)$. Moreover, the current iteration index k will remain implicit whenever unambiguous: typically, the stepsize α_k^i will be denoted by α^i. Using such compact notations, the stepsize update (7) reads

$$\alpha^0 = 0, \tag{12a}$$

$$\alpha^{i+1} = \alpha^i - \theta \dot{f}(\alpha^i)/a_i, \quad i \in \{0, \dots, I-1\}, \tag{12b}$$

$$\alpha_k = \alpha^I, \tag{12c}$$

with $\dot{f}(\alpha^i) = d_k^{\mathrm{t}} \nabla\mathcal{J}\left(x_k + \alpha^i d_k\right)$ and $a_i = d_k^{\mathrm{t}} Q_k^i d_k$. According to (8) we have

$$0 < \nu_1 \|d_k\|^2 \leq a_i \leq \nu_2 \|d_k\|^2. \tag{13}$$

Alternatively, let the stepsize be defined by a fixed number I of iterations of Weiszfeld method. The (scalar) function to minimize is f and, according to Assumption A3,

$$q_i(\alpha', \alpha) = \widehat{\mathcal{J}}_k^i(x_k + \alpha' d_k, x_k + \alpha d_k) = f(\alpha) + (\alpha' - \alpha)\, \dot{f}(\alpha) + (\alpha' - \alpha)^2\, a_i/2 \tag{14}$$

is an upper convex quadratic approximation of $f(\alpha')$. Then, the successive iterations of the Weiszfeld method read

$$\alpha^{i+1} = \arg\min_{\alpha'} q_i(\alpha', \alpha^i) = \alpha^i - \dot{f}(\alpha^i)/a_i.$$

Hence, (12) identifies with a relaxed version of Weiszfeld method to minimize f. Note that the convergence results in Sect. 4 hold regardless of the value of I. In comparison, the convergence of usual line search procedures requires appropriate stopping conditions, e.g. the Wolfe conditions.

As already mentioned, the stepsize formula (9) of [1, 2] corresponds to one iteration of the relaxed Weiszfeld method. We are now led to a deeper result: the condition $\theta \in (0, \nu_1/\mu)$ stated in [1, 2] for the convergence of their CG method implies that our Assumption A3 holds. First, let us give an equivalent formulation for (9).

Let $\tilde{Q}_k^0 = Q_k^0/\theta$, so that (9) also reads $\alpha_k = -g_k^{\mathrm{t}} d_k / d_k^{\mathrm{t}} \tilde{Q}_k^0 d_k$. From (8) and $\theta \in (0, \nu_1/\mu)$, we deduce that

$$v^{\mathrm{t}} \tilde{Q}_k^0 v \geq \mu \|v\|^2, \quad \forall k \in \mathbb{N}, \forall v \in \mathbb{R}^n, \tag{15}$$

i.e., the spectrum of matrices $\tilde{Q}_k^0$ is bounded from below by μ. In other words, the constraint $\theta \in (0, \nu_1/\mu)$ stated in [1, 2] can be translated as a constraint on matrices Q_k^i. The following lemma shows that matrices $\tilde{Q}_k^0$ yield convex quadratic majorizing approximations in the sense of Assumption A3 (provided that N is a convex set).

Lemma 2.1 *Let Assumption A1 hold and let the lower bound ν_1 be not smaller than the Lipschitz constant μ. Let us restrict ourselves to the case where N is a convex set. Then, Assumption A3 holds, i.e. the function $\widehat{\mathcal{J}}_k^i$ defined by (11) fulfills (10) over N.*

Proof According to the descent lemma [13, Proposition A.24], we have

$$\mathcal{J}(x') - \mathcal{J}(x) - (x' - x)^{\mathrm{t}} \nabla \mathcal{J}(x) \leq \mu \, \|x' - x\|^2/2, \tag{16}$$

for any $x, x' \in \mathbb{R}^n$ if $\mathcal{J}$ is μ-$\mathcal{LC}^1$on $\mathbb{R}^n$. Actually, it is easy to check that (16) still holds for any $x, x' \in N$ if $\mathcal{J}$ is μ-$\mathcal{LC}^1$on N, provided that N is convex.

Since the spectrum of (Q_k^i) is bounded from below by $\nu_1 \geq \mu$, we have

$$\mu \|x' - x\|^2 \leq \nu_1 \|x' - x\|^2 \leq (x' - x)^{\mathrm{t}} Q_k^i (x' - x).$$

Jointly with (16), the latter yields

$$\mathcal{J}(x') - \mathcal{J}(x) + (x - x')^{\mathrm{t}} \nabla \mathcal{J}(x) \leq (x' - x)^{\mathrm{t}} Q_k^i (x' - x)/2,$$

i.e., $\widehat{\mathcal{J}}_k^i(x', x) \geq \mathcal{J}(x')$. □

Lemma 2.1 indicates that Assumption A3 is not a restrictive condition compared to the hypotheses found in [1, 2]. On the contrary, it is a weaker assumption (let alone the fact that Lemma 2.1 only applies when N is convex), so that a convergence proof based on Assumption A3 would be of broader applicability. This is the goal reached in Sect. 4, where $\mathcal{J}$ is not necessarily assumed ν_1-$\mathcal{LC}^1$ (and N is not necessarily convex).

3 Properties of the Stepsize Series

The present section gathers technical results concerning the stepsize series $\alpha^i = \alpha_k^i$ generated by (7), which will be useful to derive the global convergence properties of the next section. Remark that Assumption A2 is never used in this section.

Let us introduce the notation $\Gamma(a,b)=[\min(a,b),\max(a,b)]$ to handle with intervals with unordered endpoints.

Lemma 3.1 *Let Assumptions A1 and A3 hold and let $\theta\in(0,2)$. Then,*

$$\mathcal{J}(x_k+\alpha d_k)\le\mathcal{J}(x_k^i),\quad \forall\alpha\in\Gamma(\alpha_k^i,\alpha_k^{i+1}) \tag{17}$$

for all $k\ge 0$, $i\in\{0,\dots,I-1\}$, where $x_k^i=x_k+\alpha_k^i d_k$.

Proof Let us first assume $x_k^0\in N$, and let us show that (17) holds, recursively on i. $\dot f(\alpha_k^i)=d_k^{\mathrm{t}}\nabla\mathcal{J}(x_k+\alpha_k^i d_k)$ exists since $\mathcal{J}$ is differentiable on N. We have $\alpha_k^0=0$ and $\dot f(0)=d_k^{\mathrm{t}}g_k\le 0$, but the sign of $\dot f(\alpha_k^i)$ is indeterminate for $i>0$. Let us study each case separately (the index k is omitted in the rest of the proof).

- Suppose $\dot f(\alpha^i)=0$. According to (12), $\alpha^{i+1}=\alpha^i$ so (17) is true.
- Suppose $\dot f(\alpha^i)<0$. According to (12) and $a_i>0$ we have $\alpha^{i+1}>\alpha^i$. Let us prove (17) by contradiction: suppose, on the contrary, that there exists $\alpha'\in(\alpha^i,\alpha^{i+1}]$ such that

$$f(\alpha')>f(\alpha^i). \tag{18}$$

  Let $\ell^i=\{\alpha\in\mathbb{R}|f(\alpha)\le f(\alpha^i)\}$. Since f is continuous on ℓ^i, according to (18) and $\dot f(\alpha^i)<0$, there exists $\alpha''\in(\alpha^i,\alpha')$ such that $f(\alpha'')<f(\alpha^i)$. There also exists $\alpha'''\in(\alpha'',\alpha')$ such that

$$f(\alpha''')=f(\alpha^i); \tag{19}$$

  otherwise, since f is continuous on ℓ^i, the inequality $f(\alpha)<f(\alpha^i)$ would hold for all $\alpha\in(\alpha'',\alpha')$. In particular, we would get $\lim_{\alpha\to(\alpha')^-}f(\alpha)<f(\alpha^i)$, so (α'',α') would be included into ℓ^i, which is incompatible with (18) given the continuity of f on ℓ^i.

  Now let $q(\alpha)=q_i(\alpha,\alpha^i)$, where q_i is defined by (14). Since $\dot q(\alpha^{i+1})=\dot f(\alpha^i)\times(1-\theta)$, $\dot q(\alpha^i)=\dot f(\alpha^i)<0$ and $\theta\in(0,2)$, we have $\dot q(\alpha^{i+1})\in(\dot q(\alpha^i),-\dot q(\alpha^i))$. Because q is a convex parabola and $\alpha'''\in(\alpha'',\alpha')\subset(\alpha^i,\alpha^{i+1})$, we can conclude that $q(\alpha''')<q(\alpha^i)=f(\alpha^i)$. Hence, according to (19), we get $q(\alpha''')<f(\alpha''')$, which contradicts the majorizing character (10) of $\widehat{\mathcal{J}}_k^i$ w.r.t. $\mathcal{J}$ at $x_k+\alpha''' d_k\in N$.
- Suppose that $\dot f(\alpha^i)>0$. According to (12) and $a_i>0$, we have $\alpha^{i+1}<\alpha^i$. We are led back to the previous case if we replace $f(\alpha)$ by $f(-\alpha)$.

As a first conclusion, (17) holds for all $i\in\{0,\dots,I-1\}$. Hence, $\mathcal{J}(x_k^{i+1})\le\mathcal{J}(x_k^i)$.

Since $x_0\in N$ and $x_{k+1}^0=x_k^I=x_k$, we get by recursion

$$\mathcal{J}(x_k^0)=\mathcal{J}(x_{k-1}^I)\le\cdots\le\mathcal{J}(x_{k-1}^0)\le\cdots\le\mathcal{J}(x_1^0)=\mathcal{J}(x_0),\quad\forall k.$$

Hence, $x_k^0\in N$, which proves that (17) holds for all $k\ge 0$, $i\in\{0,\dots,I-1\}$. □

An immediate consequence of Lemma 3.1 is

$$x_k + \alpha d_k \in N, \quad \forall \alpha \in [0, \alpha_k^i], \tag{20}$$

for all $k \geq 0$, $i \in \{0, \dots, I-1\}$ since $x_0 \in N$. Thus, according to (10),

$$q_i(\alpha^j, \alpha^i) \geq f(\alpha^j), \quad \forall i, j \in \{0, \dots, I-1\}. \tag{21}$$

The following three lemmas are specific to the case when $\dot{f}(0) = g_k^t d_k$ does not vanish for the current iteration k, i.e., $g_k^t d_k < 0$. Then $d_k \neq 0$, and the series (α_k^i) is well defined according to (7b).

Lemma 3.2 *Let Assumptions A1 and A3 hold. Let also $\dot{f}(0) < 0$ and $\theta \in (0, 2)$. Then, the whole series (α^i) is positive,*

$$\alpha^i > 0, \quad \forall i \in \{0, \dots, I-1\}. \tag{22}$$

Proof According to (21), we have

$$q_i(0, \alpha^i) = f(\alpha^i) - \alpha^i \dot{f}(\alpha^i) + (\alpha^i)^2 a_i/2 \geq f(0).$$

Since $(f(\alpha^i))$ is a nonincreasing series according to Lemma 3.1, we deduce that

$$-\alpha^i\, \dot{f}(\alpha^i) + (\alpha^i)^2\, a_i/2 \geq 0$$

so that, according to (12) and $a_i > 0$,

$$\alpha^i\, (\alpha^{i+1} - 2\delta \dot{f}(\alpha^i)/a_i) \geq 0, \tag{23}$$

with

$$\delta = 1 - \theta/2 \in (0, 1). \tag{24}$$

Now, let us show (22) by recurrence on i. We have $\alpha^1 > 0$ according to (12) and $\dot{f}(0) < 0$. Let us assume $\alpha^i > 0$ for some i. If $\dot{f}(\alpha^i) \leq 0$, then $\alpha^{i+1} > 0$ according to (12). If $\dot{f}(\alpha^i) > 0$, then given $\alpha^i > 0$, inequality (23) yields $\alpha^{i+1} > 0$. □

Lemma 3.3 *Let Assumptions A1 and A3 hold. Let also $\dot{f}(0) < 0$ and $\theta \in (0, 2)$. Then,*

$$f(\alpha^i) \leq q_0(\alpha^1, 0), \tag{25}$$

$$c^{\min} \alpha^1 \leq \alpha^i, \tag{26}$$

for all $i \in \mathbb{N} - \{0\}$, where

$$c^{\min} = (\sqrt{1 + 2\mu\theta\delta/\nu_1} - 1)\nu_1/\theta\mu \in (0, 1). \tag{27}$$

Proof According to (21), we have $q_0(\alpha^1, 0) \geq f(\alpha^1)$. Then (25) holds, because $(f(\alpha^i))$ is a decreasing series according to Lemma 3.1.

The derivation of (26) is not so direct. Let g be the concave parabola defined by

$$g(\alpha) = f(0) + \alpha \dot{f}(0) - \mu a_0 \alpha^2 / 2\nu_1. \tag{28}$$

Remark that $g(0) = f(0)$ and that g is decreasing on $\mathbb{R}^+$, since $\dot{g}(0) = \dot{f}(0) < 0$.

Let us first show that

$$g(\alpha^i) \leq f(\alpha^i). \tag{29}$$

Let us consider $\alpha \in [0, \alpha^i]$: $x_k + \alpha d_k \in N$ according to (20). Since $f(\alpha) = \mathcal{J}(x_k + \alpha d_k)$ and Assumption A1 holds, we have

$$|\dot{f}(\alpha) - \dot{f}(0)| = \left| d_k^{\mathrm{t}} (\nabla \mathcal{J}(x_k + \alpha d_k) - \nabla \mathcal{J}(x_k)) \right| \leq \|d_k\|^2 \mu |\alpha|$$

and according to (13), we get $|\dot{f}(\alpha) - \dot{f}(0)| \leq a_0 \mu \alpha / \nu_1$. Given $|\dot{f}(\alpha)| \leq |\dot{f}(\alpha) - \dot{f}(0)| + |\dot{f}(0)|$ and $\dot{f}(0) < 0$, we obtain

$$|\dot{f}(\alpha)| \leq a_0 \mu \alpha / \nu_1 - \dot{f}(0). \tag{30}$$

Thus, $\dot{f}(0) - a_0 \mu \alpha / \nu_1 \leq \dot{f}(\alpha)$, or equivalently

$$\dot{g}(\alpha) \leq \dot{f}(\alpha), \quad \forall \alpha \in [0, \alpha^i], \tag{31}$$

according to (28). Since $g(0) = f(0)$, integrating (31) between 0 and α^i yields (29).

According to (9), (12), (14) and (24), we have

$$\begin{aligned} q_0(\alpha^1, 0) &= f(0) + \alpha^1 \dot{f}(0) + (\alpha^1)^2 a_0 / 2 \\ &= f(0) + \delta \alpha^1 \dot{f}(0). \end{aligned} \tag{32}$$

Then let us show that $q_0(\alpha^1, 0) = g(\alpha^{\min})$, where $\alpha^{\min} = c^{\min} \alpha^1$. From (27) we have

$$\begin{aligned} (c^{\min})^2 &= (2 + 2\mu\theta\delta/\nu_1 - 2\sqrt{1 + 2\mu\theta\delta/\nu_1})\nu_1^2 / (\theta\mu)^2 \\ &= (\delta - c^{\min}) 2\nu_1 / \theta\mu. \end{aligned} \tag{33}$$

According to (28), we have also

$$\begin{aligned} g(\alpha^{\min}) &= f(0) + c^{\min} \alpha^1 \dot{f}(0) - \mu a_0 (c^{\min} \alpha^1)^2 / 2\nu_1 \\ &= f(0) + \alpha^1 \dot{f}(0)(c^{\min} + (c^{\min})^2 \theta\mu / 2\nu_1) \end{aligned} \tag{34}$$

according to $\alpha^1 = -\theta \dot{f}(0)/a_0$. Jointly with (33), (34) yields $g(\alpha^{\min}) = f(0) + \delta\alpha^1 \dot{f}(0)$, so that $g(\alpha^{\min})$ identifies with $q_0(\alpha^1, 0)$ according to (32).

On the other hand, (α^i) is positive according to Lemma 3.2. We are now in position to show (26) by contradiction: assume that there exists $i > 0$ such that $0 \leq \alpha^i < \alpha^{\min}$. According to (29) and given that g is decreasing on $\mathbb{R}^+$, we get $f(\alpha^i) \geq g(\alpha^i) > g(\alpha^{\min}) = q_0(\alpha^1, 0)$, which contradicts (25).

Finally, it is obvious that $0 < c^{\min} < \delta < 1$ from the alternative expression

$$c^{\min} = 2\delta/(\sqrt{1 + 2\mu\theta\delta/\nu_1} + 1). \qquad \square$$

Lemma 3.4 *Let Assumptions A1 and A3 hold. Let also $\dot{f}(0) < 0$ and $\theta \in (0, 2)$. Then,*

$$\alpha^i \le c_i^{\max}\alpha^1, \quad \forall i \in \mathbb{N} - \{0\}, \tag{35}$$

with

$$c_i^{\max} = (1 + \nu_2\theta\mu/\nu_1^2)^{i-1}(1 + \nu_1/\theta\mu) - \nu_1/\theta\mu \ge 1. \tag{36}$$

Proof It is easy to check that $c_i^{\max}$ is not smaller than 1 for all $i > 0$. Let us show the inequality (35) recursively on i. It is valid for $i = 1$, since $c_1^{\max} = 1$. Now let us suppose that $\alpha^i \le c_i^{\max}\alpha^1$, and let us prove that $\alpha^{i+1} \le c_{i+1}^{\max}\alpha^1$.

According to (12), we have $\alpha^{i+1} \le \alpha^i + |\dot{f}(\alpha^i)|\theta/a_i$ and according to (13), we have also $a_i \ge a_0\nu_1/\nu_2$. Thus,

$$\alpha^{i+1} \le \alpha^i + |\dot{f}(\alpha^i)|\theta\nu_2/\nu_1 a_0. \tag{37}$$

On the other hand, (30) implies $|\dot{f}(\alpha^i)| \le a_0\mu\alpha^i/\nu_1 - \dot{f}(0)$. In combination with the latter inequality and with $\alpha^1 = -\theta\dot{f}(0)/a_0$, (37) yields

$$\alpha^{i+1} \le \alpha^i(1 + \nu_2\theta\mu/\nu_1^2) + \nu_2\alpha^1/\nu_1,$$

which corresponds to a recursive definition of the series $(c_i^{\max})$ according to

$$c_{i+1}^{\max} = c_i^{\max}(1 + \nu_2\theta\mu/\nu_1^2) + \nu_2/\nu_1.$$

Given $c_1^{\max} = 1$, it can be checked that (36) is the general term of the series. $\square$

Definition 3.1 The stepsize series (α_k) satisfies the Armijo condition for $\Omega \in (0, 1)$ if

$$\mathcal{J}(x_k) - \mathcal{J}(x_{k+1}) + \Omega\alpha_k g_k^{\mathrm{t}} d_k \ge 0, \quad \forall k. \tag{38}$$

Lemma 3.5 *Let Assumptions A1 and A3 hold. Let also $\theta \in (0, 2)$. Then, the stepsize series defined by (7) satisfies the Armijo condition for*

$$\Omega = \Omega_I = \delta/c_I^{\max} \in (0, 1), \tag{39}$$

where δ and $c_I^{\max}$ are defined by (24) and (36), respectively.

Proof We have $\dot{f}(0) = g_k^{\mathrm{t}} d_k \le 0$. Let us first examine the particular case $\dot{f}(0) = 0$: according to (7), α_k vanishes, so that (38) holds trivially.

Suppose now that $\dot{f}(0) < 0$. According to (32), (25) also reads

$$f(0) - f(\alpha^I) + \delta\dot{f}(0)\alpha^1 \ge 0. \tag{40}$$

Finally, since $\dot{f}(0)<0$ and $\alpha^1 \geq \alpha^I/c_I^{\max}>0$ according to (35), (40) implies that

$$f(0)-f(\alpha^I)+\delta \dot{f}(0)\alpha^I/c_I^{\max} \geq 0,$$

which identifies with (38) with $\Omega=\Omega_I$. □

Remark 3.1 In Lemma 3.5, $\Omega_I=\delta/c_I^{\max}$ does not depend on k, which is an essential point for the fulfillment of the Armijo condition.

The following theorem sums up the main results that will be useful in the next section.

Theorem 3.1 *Let x_k be defined by (1–7) with $\theta \in (0,2)$, and let Assumptions A1 and A3 hold. Then, the Armijo condition (38) is satisfied by the stepsize series (α_k) for $\Omega=\Omega_I=\delta/c_I^{\max}$, where δ and $c_I^{\max}$ are defined by (24) and (36) respectively. Moreover, we have*

$$0 \leq c^{\min}\alpha_k^1 \leq \alpha_k \leq c_I^{\max}\alpha_k^1, \quad \forall k, \tag{41}$$

where $c^{\min}$ is defined by (27).

Proof Lemma 3.5 corresponds to the fulfillment of the Armijo condition.

On the other hand, we have $\dot{f}(0) \leq 0$. If $\dot{f}(0)=0$, then $\alpha_k=0$, so (41) trivially holds. Otherwise, we have $\dot{f}(0)<0$, so (41) is a joint consequence of Lemmas 3.3 and 3.4. □

4 Global Convergence

The following two lemmas establish results for the whole two-parameter family of conjugacy coefficients $\beta_k=\beta_k^{\mu_k,\omega_k}$. Then, we draw conclusions for specific CG methods.

Lemma 4.1 *Under the conditions of Theorem 3.1, we have*

$$\sum_{k,\, d_k\neq 0} (g_k^{\mathrm{t}} d_k)^2/\|d_k\|^2 < \infty. \tag{42}$$

Proof According to Theorem 3.1, the Armijo condition (38) is satisfied for $\Omega=\Omega_I$. Given (41) and $g_k^{\mathrm{t}} d_k \leq 0$, we deduce that

$$\mathcal{J}(x_k)-\mathcal{J}(x_{k+1}) \geq -\Omega_I c^{\min}\alpha_k^1 g_k^{\mathrm{t}} d_k. \tag{43}$$

If $d_k \neq 0$, we have

$$\alpha_k^1=-\theta g_k^{\mathrm{t}} d_k/d_k^{\mathrm{t}} Q_k^0 d_k \geq -\theta g_k^{\mathrm{t}} d_k/\nu_2\|d_k\|^2, \tag{44}$$

according to (9) and (8), so that

$$\mathcal{J}(x_k) - \mathcal{J}(x_{k+1}) \geq c_0 (g_k^t d_k)^2 / \|d_k\|^2 \geq 0, \tag{45}$$

with $c_0 = \Omega_I c^{\min} \theta / \nu_2 > 0$. Given Assumption A1 and since L is assumed bounded, (45) implies that $\lim_{k\to\infty} \mathcal{J}(x_k)$ is finite. Finally, we obtain

$$\infty > (\mathcal{J}(x_0) - \lim_{k\to\infty} \mathcal{J}(x_k))/c_0 \geq \sum_{k,\, d_k \neq 0} (g_k^t d_k)^2 / \|d_k\|^2.$$

□

Lemma 4.2 *Let $k \in \mathbb{N}$. Under the conditions of Theorem 3.1, we have*

$$|g_{k+1}^t d_k| \leq -g_k^t d_k (1 + c_I^{\max} \theta \mu / \nu_1). \tag{46}$$

Moreover, if Assumption A2 holds, then

$$-g_{k+1}^t d_k \leq -g_k^t d_k (1 - c^{\min} \theta \lambda / \nu_2). \tag{47}$$

Proof Equations (46) and (47) are trivial assertions if $d_k = 0$. Otherwise, following [1, 2], let us define

$$\phi_k = \begin{cases} y_k^t d_k / \alpha_k \|d_k\|^2, & \alpha_k \neq 0, \\ 0, & \alpha_k = 0. \end{cases} \tag{48}$$

Note that according to (20), $x_k \in N$. If Assumption A1 holds, then $|\phi_k| \leq \mu$ according to Cauchy-Schwartz inequality. If Assumption A2 holds, then $\phi_k \geq \lambda > 0$.

According to (9) and (48), we have

$$g_{k+1}^t d_k = g_k^t d_k + y_k^t d_k = g_k^t d_k + \alpha_k \phi_k \|d_k\|^2. \tag{49}$$

According to (41), (49), $\mu \geq |\phi_k|$, and $g_k^t d_k \leq 0$, we deduce that

$$|g_{k+1}^t d_k| \leq -g_k^t d_k + \mu c_I^{\max} \alpha_k^1 \|d_k\|^2.$$

According to (9), we have also

$$|g_{k+1}^t d_k| \leq -g_k^t d_k - g_k^t d_k \mu c_I^{\max} \theta \|d_k\|^2 / d_k^t Q_k^0 d_k.$$

Finally, since $\nu_1 > 0$ is a lower bound for the spectrum of Q_k^0, and $g_k^t d_k \leq 0$, we obtain (46). Let us suppose now that Assumption A2 holds. Given (41) and $\phi_k \geq \lambda > 0$, (49) implies $g_{k+1}^t d_k \geq g_k^t d_k + \lambda c^{\min} \alpha_k^1 \|d_k\|^2$ and, according to (44), we obtain (47). □

Lemma 4.3 *Let Assumption A2 hold, together with the conditions of Theorem 3.1. Then,*

$$D_k \geq (1 - \mu_k - \omega_k)\|g_{k-1}\|^2 - d_{k-1}^t g_{k-1}(\omega_k + \mu_k c^{\min} \theta \lambda / \nu_2) \geq 0, \quad \forall k \in \mathbb{N} - \{0\}. \tag{50}$$

Proof Since $y_{k-1} = g_k - g_{k-1}$, (47) also reads

$$d_{k-1}^{\mathrm{t}} y_{k-1} \geq -d_{k-1}^{\mathrm{t}} g_{k-1} c^{\min} \theta\lambda/\nu_2, \quad \forall k \in \mathbb{N} - \{0\}.$$

Then, given the expression (5) of D_k and $d_{k-1}^{\mathrm{t}} g_{k-1} \leq 0$, the conclusion is immediate. □

Remark 4.1 Let us examine the case where the denominator D_k of $\beta_k^{\mu_k,\omega_k}$ vanishes. Here, we assume that the conditions of Theorem 3.1 hold.

Let us suppose first that Assumption A2 is valid. If D_k vanishes, then (50) implies

$$(1 - \mu_k - \omega_k)\|g_{k-1}\|^2 - (\omega_k + \mu_k c^{\min} \theta\lambda/\nu_2) d_{k-1}^{\mathrm{t}} g_{k-1} = 0.$$

Since the left-hand side is the sum of two nonnegative terms, we obtain

$$(1 - \mu_k - \omega_k)\|g_{k-1}\|^2 = 0, \tag{51a}$$

$$(\omega_k + \mu_k c^{\min} \theta\lambda/\nu_2) d_{k-1}^{\mathrm{t}} g_{k-1} = 0. \tag{51b}$$

- Case 1: If $\mu_k + \omega_k < 1$, (51a) reduces to $\|g_{k-1}\|^2 = 0$, which means that convergence is reached at iteration $k - 1$. This case includes the PRP method.
- Case 2: If $\mu_k + \omega_k = 1$, (51b) implies $d_{k-1}^{\mathrm{t}} g_{k-1} = 0$, so that $\alpha_{k-1} = 0$. Thus, $x_k = x_{k-1}$, $y_{k-1} = 0$, and the numerator of $\beta_k^{\mu_k,\omega_k}$ vanishes. In this case, we let $\beta_k^{\mu_k,\omega_k} = 0$, conventionally. This case includes the HS and the LS method.

In the situation where Assumption A2 is not necessarily valid, our study covers only the case $\mu_k = 0$: then D_k is the sum of two nonnegative terms, so $D_k = 0$ implies that both vanish:

$$(1 - \omega_k)\|g_{k-1}\|^2 = 0 \quad \text{and} \quad \omega_k d_{k-1}^{\mathrm{t}} g_{k-1} = 0.$$

- If $\omega_k < 1$, the conclusion is the same as in Case 1. This case includes the PRP method.
- If $\omega_k = 1$, the conclusion is the same as in Case 2. This case includes the LS method.

Lemma 4.4 *Under the conditions of Theorem* 3.1, *we have*

$$\liminf_{k\to\infty} \|g_k\| > 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \beta_k^{0,\omega_k} = 0.$$

Moreover, if Assumption A2 is valid, then

$$\liminf_{k\to\infty} \|g_k\| > 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \beta_k^{\mu_k,\omega_k} = 0.$$

Proof According to (1) and (41), we have

$$\|x_{k+1} - x_k\|^2 = \alpha_k^2 \|d_k\|^2 \leq (c_I^{\max})^2 (\alpha_k^1)^2 \|d_k\|^2.$$

Given that (9) holds unless $d_k = 0$, we deduce that

$$\sum_k \|x_{k+1} - x_k\|^2 \le (c_I^{\max}\theta)^2 \sum_{k,\, d_k \ne 0} (g_k^t d_k)^2 \|d_k\|^2/(d_k^t Q_k^0 d_k)^2,$$
$$\le (c_I^{\max}\theta/\nu_1)^2 \sum_{k,\, d_k \ne 0} (g_k^t d_k)^2/\|d_k\|^2,$$

according to (8). Given (42), we conclude that $\lim_{k\to\infty} \|x_{k+1} - x_k\|^2 = 0$. Because $\mathcal{J}$ is continuously differentiable and $\|g_k\|$ is bounded according to Assumption A1 and the boundedness of L, we have also $\lim_{k\to\infty} y_{k-1} = 0$ and

$$\lim_{k\to\infty} g_k^t y_{k-1} = 0. \tag{52}$$

If $\liminf_{k\to\infty} \|g_k\| > 0$, then there exists $\gamma > 0$ such that

$$\|g_k\| \ge \gamma > 0, \quad \forall k. \tag{53}$$

According to (4), we have

$$|g_k^t y_{k-1}| = |\beta_k^{\mu_k,\omega_k}||D_k|. \tag{54}$$

On the one hand, suppose that Assumption A2 is valid.

First, let us consider the iteration indices k such that $\mu_k + \omega_k \in [0, 1/2]$. According to (50) and $d_{k-1}^t g_{k-1} \le 0$, (54) implies that

$$|g_k^t y_{k-1}| \ge |\beta_k^{\mu_k,\omega_k}|(1 - \mu_k - \omega_k)\|g_{k-1}\|^2.$$

Given (53), the latter inequality yields

$$|g_k^t y_{k-1}| \ge |\beta_k^{\mu_k,\omega_k}|\gamma^2/2. \tag{55}$$

Let us establish a similar result in the more complex case $\mu_k + \omega_k \in (1/2, 1]$. As a preliminary step, let us show that

$$g_k^t d_k \le -\gamma^2/2 \tag{56}$$

for all sufficiently large values of k.

According to Remark 4.1, in the case $g_{k-1}^t d_{k-1} = 0$, we have $\beta_k^{\mu_k,\omega_k} = 0$, so $d_k = -g_k$ and (56) is valid according to (53).

Now, let us consider the case where $g_{k-1}^t d_{k-1} < 0$. Given (2) and (4), we have

$$g_k^t c_k = g_k^t(-g_k + \beta_k^{\mu_k,\omega_k} d_{k-1}) = -\|g_k\|^2 + (g_k^t y_{k-1})(g_k^t d_{k-1})/D_k.$$

Given $\mu_k + \omega_k \in (1/2, 1]$, we have $\omega_k + \mu_k c^{\min}\theta\lambda/\nu_2 \ge m$, where $m = \min\{1/2, c^{\min}\theta\lambda/\nu_2\}$. As a consequence, (50) implies that $D_k \ge -m\, d_{k-1}^t g_{k-1}$. Jointly with (46) and (53), the latter inequality yields

$$g_k^t c_k \le -\gamma^2 + |g_k^t y_{k-1}|(1 + c_I^{\max}\theta\mu/\nu_1)/m.$$

Given (52), we deduce that $g_k^{\mathrm{t}} c_k \leq -\gamma^2/2$ for all sufficiently large k. Because of (3), we can conclude that (56) holds.

Given (50) and (56), (54) implies

$$\begin{aligned}|g_k^{\mathrm{t}} y_{k-1}| &\geq |\beta_k^{\mu_k,\omega_k}|((1-\mu_k-\omega_k)\gamma^2 + (\omega_k + \mu_k c^{\min}\theta\lambda/\nu_2)\gamma^2/2) \\ &= |\beta_k^{\mu_k,\omega_k}|(1-\omega_k/2-(1-c^{\min}\theta\lambda/\nu_2)\mu_k)\gamma^2,\end{aligned}$$

for all sufficiently large values of k. Given $\mu_k + \omega_k \in (1/2, 1]$, the latter inequality implies

$$|g_k^{\mathrm{t}} y_{k-1}| \geq |\beta_k^{\mu_k,\omega_k}| m\gamma^2. \tag{57}$$

Since $m \leq 1/2$, (57) is implied by (55), so that (57) holds for all $\mu_k \in [0, 1]$, $\omega_k \in [0, 1-\mu_k]$. Finally, (52) and (57) jointly imply $\lim_{k\to\infty} |\beta_k^{\mu_k,\omega_k}| = 0$.

On the other hand, consider the case where Assumption A2 is not necessarily valid. If $\mu_k = 0$, then we have $|g_k^{\mathrm{t}} y_{k-1}| \geq |\beta_k^{0,\omega_k}|\gamma^2/2$. The proof is similar to that of (57), where the two cases to examine are $\omega_k \in [0, 1/2]$ and $\omega_k \in (1/2, 1]$. Finally, according to (52), we have $\lim_{k\to\infty} |\beta_k^{0,\omega_k}| = 0$. □

Remark 4.2 The proof of Lemma 4.4 is inspired from that of [2, Lemma 3.2], but we deal with the more general case of the iterated formula (7). Moreover, μ_k and ω_k are possibly varying, while they are constant parameters in [2].

Theorem 4.1 *Let x_k be defined by* (1–7) *with* $\theta \in (0, 2)$, *and let Assumptions A1 and A3 hold. Then, we have convergence in the sense* $\liminf_{k\to\infty} g_k = 0$ *for the PRP and LS methods, and more generally for* $\mu_k = 0$ *and* $\omega_k \in [0, 1]$. *Moreover, if Assumption A2 holds, then we have also* $\liminf_{k\to\infty} g_k = 0$ *in all cases.*

Proof Assume on the contrary that $\|g_k\| \geq \gamma > 0$ for all k. Since L is bounded, both (x_k) and (g_k) are bounded.

Let us first suppose that Assumption A2 holds. Since $\liminf_{k\to\infty} \|g_k\| > 0$, by Lemma 4.4 we have $\lim_{k\to\infty} \beta_k^{\mu_k,\omega_k} = 0$.

Since $\|d_k\| = \|c_k\| \leq \|g_k\| + |\beta_k^{\mu_k,\omega_k}| \, \|d_{k-1}\|$, we conclude that $(\|d_k\|)$ is uniformly bounded for sufficiently large k. Thus, we have

$$|g_k^{\mathrm{t}} d_k| = |g_k^{\mathrm{t}}(-g_k + \beta_k^{\mu_k,\omega_k} d_{k-1})| \geq \|g_k\|^2 - |\beta_k^{\mu_k,\omega_k}| \, \|g_k\| \, \|d_{k-1}\| \geq \|g_k\|^2/2$$

for sufficient large k. Then, there exists $\epsilon > 0$ so that $g_k^{\mathrm{t}} d_k/\|d_k\| \, \|g_k\| \geq \|g_k\|/2\|d_k\| \geq \epsilon$ for sufficient large k. Finally, we conclude that $\sum_{k,\, d_k\neq 0} \|g_k\|^2 (g_k^{\mathrm{t}} d_k/\|d_k\|\|g_k\|)^2 = \infty$, which contradicts Lemma 4.1.

The same proof applies to the case $\mu_k = 0$, whatever the validity of Assumption A2. □

Remark 4.3 The proof of Theorem 4.1 is partly inspired from that of [2, Theorem 3.3]. However, our result deals with variable parameters μ_k, ω_k. Moreover, Assumption A2 is not necessary in the case $\mu_k = 0$, which contains the PRP and LS methods.

5 Discussion

5.1 Convex Quadratic Case

Let us show that the convergence condition $\theta \in (0, \nu_1/\mu)$ is too restrictive for the stepsize formula proposed in [1, 2], in the case of a convex quadratic objective function. Let $\mathcal{Q}(x) = x^{\mathrm{t}} Q x/2 - b^{\mathrm{t}} x$, where $x \in \mathbb{R}^n$ and Q is a SPD matrix. Let ν_1 and ν_2 respectively denote the smallest and largest eigenvalue of Q, so that (8) holds. Now, consider the stepsize formula (9) with $Q_k^0 = Q$. When $\theta = 1$, it yields the optimal stepsize $\alpha_k = \arg\min_\alpha \mathcal{J}(x_k + \alpha d_k)$. In the convex quadratic case, Theorem 4.1 ensures the convergence for $\theta \in (0, 2)$ and for any fixed $I > 0$. Remark that Assumption A3 is easily checked, since

$$\hat{\mathcal{Q}}(x', x) = \mathcal{Q}(x) + (x' - x)^{\mathrm{t}}\, \nabla\mathcal{Q}(x) + (x' - x)^{\mathrm{t}}\, Q\,(x' - x)/2 = \mathcal{Q}(x').$$

In the case of the optimal stepsize, i.e., $\theta = 1$, the classical linear CG algorithm is covered. On the contrary, the convergence domain in [1, 2] is reduced to $\theta \in (0, \nu_1/\nu_2) \subset (0, 1)$, since $\nabla\mathcal{Q}$ is μ-$\mathcal{LC}^1$with $\mu \leq \nu_2$. Hence, the convergence of the linear CG algorithm is not covered. Moreover, the condition $\theta < \nu_1/\nu_2$ will produce excessively small and inefficient stepsizes when Q is ill-conditioned.

5.2 General Case

In principle, the nontrivial computation of ν_1 and μ is a prerequisite to check the convergence condition $\theta \in (0, \nu_1/\mu)$. In [2], it is rather proposed to ensure the convergence empirically by choosing arbitrarily small values of θ. However, the resulting algorithm will be hardly competitive, compared to CG methods with a usual line search procedure.

Our convergence results do not share the same drawback, provided that, as a preliminary step, a convex quadratic function has been found to approximate the objective function from above. According to Lemma 2.1, finding such a convex quadratic majorizing function is always possible when N is a convex set.

In practice, case-by-case considerations may provide tighter convex quadratic approximations, that will result in larger stepsizes. This issue is actually not new, since finding a good convex quadratic majorizing function is already a crucial step in the use of Weiszfeld method [12]. The latter reference provides examples in the field of optimal location (which was Weiszfeld's original concern), and in structural mechanics. Robust regression is another area where Weiszfeld method is widely applied, under the name of *Iterative Reweighting* [14]. More recently, it has also become a standard approach for edge preserving image restoration, under the name of *Half-Quadratic Scheme* [15, 16].

5.3 Edge-Preserving Image Restoration

Edge preserving image restoration is structurally close to robust regression [14], but it is of much larger scale since digital images commonly gather billions of pixels. In typical image restoration problems, a sought image $\hat{x} \in \mathbb{R}^n$ is estimated from a noisy, blurred version $y \in \mathbb{R}^p$. Consider the linear observation model $y = Ax + \varepsilon$, where $A \in \mathbb{R}^{p\times n}$ is a Toeplitz-block-Toeplitz matrix that represents the observation system, while ε gathers measurement uncertainty and any other source of errors in the data. The sought image is customarily estimated as the minimizer of the following penalized least square (PLS) function, incorporating an edge-preserving penalization term [15–17]:

$$\mathcal{J}(x) = \frac{1}{2}\,\|Ax - y\|^2 + \lambda \sum_{c=1}^{C} \phi([Vx]_c), \tag{58}$$

where $\lambda > 0$ is a regularization parameter and V is a first-order difference matrix. $\phi_{\text{hyp}}(u) = \sqrt{\delta^2 + |u|^2}$ is a typical example of a convex function that preserves edges better than a quadratic. In the image restoration example considered here, matrix A stands for a blurring operation involving a Gaussian point spread function. The choice $\phi = \phi_{\text{hyp}}$ ensures that $\mathcal{J}$ (58) is then coercive and strictly convex.

The minimization of PLS functions (58) is customarily addressed using the Geman and Reynolds (GR) algorithm [15, 17], which is shown to fall within the class of Weiszfeld algorithms [16]. The GR algorithm can be defined by

$$\begin{aligned} x^{k+1} &= x^k + d^k_{\text{GR}},\\ d^k_{\text{GR}} &= -(Q_{\text{GR}}(x^k))^{-1} g_k, \end{aligned} \tag{59}$$

where the normal operator $Q_{\text{GR}}(u)$ is defined by

$$\begin{aligned} Q_{\text{GR}}(u) &= A^{\text{t}}A + \lambda V^{\text{t}}\,\text{Diag}\{b(u)\}V,\\ b(u) &= \text{Vect}[\phi'([Vu]_c)/[Vu]_c]. \end{aligned}$$

The inversion of the linear system (59) is required to determine the descent direction. In the case of large-scale problems, as typically encountered in image restoration, the resulting numerical cost is generally too high. In practice, it is rather proposed to compute an inexact descent direction using a truncated linear CG (TLCG) method [15, 17].

The main goal of this section is to experimentally compare the following three schemes:

- CG+GR1D(I) refers to the algorithm defined by (1–7) with $\beta_k = \beta_k^{\text{PRP}}$, $\theta = 1$,

$$Q_k^i = Q_{\text{GR}}(x^k + \alpha_k^i d^k), \tag{60}$$

  and where I is the fixed number of iterations of (7b). Note that both Assumptions A1 and A3 hold on $N = \mathbb{R}^n$ according to [16].
- CG+SWOLFE(c_1, c_2) refers to the algorithm defined by (1–7) with $\beta_k = \beta_k^{\text{PRP}}$, $\theta = 1$ and where the stepsize is computed according to Algorithms 3.2 and 3.3 of [18]. In particular, it satisfies the following strong Wolfe conditions

$$\begin{aligned} \mathcal{J}(x_k) - \mathcal{J}(x_{k+1}) + c_1\alpha_k g_k^{\text{t}} d_k &\geq 0,\\ \nabla\mathcal{J}(x_k + \alpha_k d_k)^{\text{t}} d_k - c_2 g_k^{\text{t}} d_k &\geq 0, \end{aligned} \tag{61}$$

Table 1 Comparison of the algorithms for the deblurring problem

	Without preconditioning		With CT preconditioning	
	Iterations	Time (s)	Iterations	Time (s)
CG+GR1D(1)	89/1/1	129.16	28/1/1	53.66
CG+GR1D(2)	93/1/2	141.62	30/1/2	59.92
CG+GR1D(5)	93/1/5	161.81	31/1/5	68.74
CG+GR1D(10)	93/1/10	196.21	31/1/10	80.13
CG+SWOLFE(10^{-4}, 0.1)	95/2.52/2.52	287.45	32/1.94/1.94	89.71
CG+SWOLFE(10^{-4}, 0.5)	96/1.76/1.77	204.96	33/1.06/1.15	59.22
CG+SWOLFE(10^{-4}, 0.9)	298/1.44/1.44	518.45	35/1/1.14	60.74
GR+TLCG(10^{-6})	12/1/102.1	1936.46	12/1/32.3	822.05

with $0 < c_1 < c_2 < 1$. Note that (61) identifies with the Armijo condition (38). The line search is initialized with the unit stepsize. Following [18, Chap. 3] we select $c_1 = 10^{-4}$, and several values of parameter c_2 are tested.

- GR+TLCG(η) refers to an approximate form of the GR algorithm using a TLCG method [17]. The TLCG is used with the stopping rule $\|r^i\|/\|r^0\| < \eta$, where r^i is the normal equation residual after i iterations. Here, we select $\eta = 10^{-6}$.

We consider also preconditioning for the three algorithms using the 2D fast cosine transform (CT) [19]. In all cases, the chosen stopping rule is $\|\nabla \mathcal{J}(x_k)\|/n < 10^{-6}$.

The experiments were undertaken under Matlab 7 on a PC P4 2.8 GHz RAM 1 Gb. The image size is $n = 512^2$. Table 1 displays iteration numbers before convergence, under the form $a/b/c$, where a is the number of global iterations and b is the average number of gradient evaluations per global iteration. The meaning of c is specific to each algorithm:

- in the case of CG+GR1D(I), c identifies with I,
- in the case of CG+SWOLFE(c_1, c_2), c is the average number of criterion evaluation per global iteration,
- in the case of GR+TLCG(η), c is the average number of TLCG iterations per global iteration.

Table 1 also displays the minimization time for each algorithm. It clearly shows that algorithm GR+TLCG(10^{-6}) is outperformed by the two CG algorithms. Actually, GR+TLCG(10^{-6}) needs fewer global iterations, but many more operations per global iteration.

The CG+GR1D(I) algorithm is shown to perform best when I is very small ($I = 1$ or $I = 2$). Increasing I does not lead to a more favorable tradeoff since the number of global iteration remains nearly constant.

The CG+SWOLFE(c_1, c_2) algorithm needs not more than three evaluations of gradient and criterion per global iteration for all of the three tested couples (c_1, c_2). The relation between (c_1, c_2) and the number of evaluations is not straightforward. However, the latter decreases when c_2 increases, because the strong Wolfe condition becomes less restrictive. At the same time, the number of global iterations increases, sometimes dramatically. Finally, the best couple (c_1, c_2) correspond to a tradeoff that is not easy to guess.

Basically similar conclusions have been reached on an image denoising problem.

In the previous comparisons, the CG+GR1D algorithm has been tested with $\theta = 1$ only. One may wonder if overrelaxed schemes $\theta \in (1, 2)$ converge faster. The answer is negative, at least in the numerical experiments that we have conducted: we have practically found that no other value of θ performs better than $\theta = 1$ for the considered problems. For the sake of compactness, the corresponding results are not reported here.

As a conclusion, the CG+GR1D(I) algorithm presents several interesting features compared to the other two tested schemes. In particular, its convergence speed is at least as high as that of CG+SWOLFE(c_1, c_2), without the tuning of any crucial parameter. It is also easier to implement, since it amounts to the application of a CFSF. As a price to pay, a preliminary study must be conducted to approximate the objective function by a convex quadratic function from above.

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