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Min-sum controllable risk problems with concave risk functions of the same value range

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Abstract

A min-sum controllable risk problem, defined on a given set of elements or on combinatorial structures, which are either paths of a directed acyclic graph or spanning trees of an undirected graph, with resource-dependent risk functions of the elements, is studied. The resource amount is limited, and the objective is to distribute it between the selected elements or elements of the selected structure so that the total risk is minimized. A reduction to a series of easier problems is suggested. Solution approaches based on this reduction are asymptotically faster than the solution approaches suggested in the literature for special cases of this problem.

KEYWORDS

bicriteria optimization, controllable data, knapsack problem, min-sum problem, risk, shortest path, spanning tree

1 | INTRODUCTION

Minimization of risks is one of the main objectives in decision making. We study risk minimization problems in network applications, where the decision comprises a subset of elements of a given set, or a path in a directed acyclic graph, or a spanning tree in an undirected graph, and the resource amounts allocated to the elements of the subset, arcs of the path, or edges of the spanning tree. The larger is the resource amount the smaller is the risk of failure of the corresponding element. The objective is to minimize the total risk.

Throughout this article we use the graph theory terminology whose basic concepts can be found, for example, in Harary [20], Bollobás [5], or Bondy and Murty [6]. Let $E=\{1, \dots, m\}$ be a set of arbitrary elements, and let $\mathcal{S}(E)$ be a specific family of nonempty subsets of E, which we call *structures*. For example, E can be the set of *arcs* of a *directed acyclic graph* $G=(V,E)$, $E \subseteq \{(i,j) \mid i \in V, j \in V\}$, where V is the set of *nodes*, and $\mathcal{S}(E)$ can be the set of all *paths* between two specified nodes of this graph. Each element $e \in E$ is associated with an integer-valued resource variable x_e, nonnegative integer lower and upper bounds c_e and d_e on this variable, $c_e < d_e$, and a *risk function* $f_e(x_e)$. We call pairs (S,x) *solutions*, where $S \in \mathcal{S}(E)$ is a structure and x is a collection of variables x_e, $e \in S$. The following assumption is employed in this article.

Assumption 1. All the risk functions f_e, $e \in E$, are nonincreasing, concave, and satisfy the equalities $f_e(c_e)=k_1$ and $f_e(d_e)=k_2$, where k_1 and k_2 are given numbers such that $k_1 > k_2$. The values k_1 and k_2 are the same for all $e \in E$.

Thus, a maximum risk value k_1 is attained for the element e, when the resource amount x_e is at its lower bound c_e, and a minimum risk value k_2 is attained when x_e is at its upper bound d_e. Let B be an integer upper bound on the total available amount of the resource. The problem studied in this article can be formulated as follows.

Problem Min-Risk.

$$\min_{(S,x)} \sum_{e \in S} f_e(x_e), \quad \text{subject to}$$

$$\sum_{e\in S} x_e \le B,$$
$$S \in \mathcal{S}(E),$$
$$x_e \in \{c_e, c_e+1, \ldots, d_e\}, \quad e \in S.$$

Our further analysis shows the validity of the following comment.

Comment 1. All the results (except for the bicriteria case) for the problem MIN-RISK, obtained in this article, apply for real-valued variables x_e, $e \in E$, and constraints $c_e \le x_e \le d_e$, $e \in E$.

The original motivation of our studies is the problem introduced by Chen et al. [9, 10]. We denote this latter problem as MIN-LINEAR-RISK.

Problem MIN-LINEAR-RISK.

$$\min_{(S,x)} \sum_{e\in S} \frac{d_e - x_e}{d_e - c_e}, \quad \text{subject to}$$
$$\sum_{e\in S} x_e \le B,$$
$$S \in \mathcal{S}(E),$$
$$x_e \in \{c_e, c_e+1, \ldots, d_e\}, \quad e \in S.$$

In Chen et al. [9, 10], E is the set of edges (or arcs) of an undirected (resp. directed acyclic) graph, and $\mathcal{S}(E)$ is the set of spanning trees (resp. paths between two specified nodes) of the corresponding graph. The authors consider function $f_e(x_e) = \frac{d_e - x_e}{d_e - c_e}$ as the value of risk of assigning weight x_e to the edge (resp. arc) e within the framework of routing and network design. In the routing setting, x_e is viewed as the time of traveling via the link e of a network, and $f_e(x_e)$ as the risk of failure of traveling via this link. The objective is to find a path between two specified nodes and to determine the travel time for each link of this path, which altogether minimize the total risk, provided that the total travel time does not exceed a given upper bound B. In the network design setting, x_e is viewed as the cost of establishing a link between the two nodes of the edge e, and $f_e(x_e)$ as the risk of failure of this link. The objective is to find a communication network, that is, a spanning tree, and to determine the cost for establishing a link for each edge of this tree, which altogether minimize the total risk, provided that the total cost does not exceed a given upper bound B. The problem MIN-LINEAR-RISK is a special case of the problem MIN-RISK. The functions $f_e(x_e) = \frac{d_e - x_e}{d_e - c_e}$ satisfy Assumption 1 with $k_1 = 1$ and $k_2 = 0$.

Risk minimization problems in network applications have recently been studied by Bayrak et al. [3], Bozkaya et al. [7], Richards et al. [32], and Le Carrer et al. [28]. Problems with controllable (resource-dependent) input parameters have been studied for scheduling problems by Cheng et al. [11, 12], Grigoriev et al. [18], Janiak and Kovalyov [21], Janiak et al. [22], Ng et al. [29], Shabtay and Steiner [33], and Shioura et al. [34].

This article is organized as follows. In the next section, we provide a reduction of the problem MIN-RISK to a series of easier problems. Section 3 presents polynomial time algorithms for several special cases of the problem MIN-RISK based on our reduction. They perform asymptotically better for all the cases in Chen et al. [9, 10]. In Section 4, a *reverse* version of the problem MIN-RISK is studied, which is denoted as REV-MIN-RISK. The reverse problem differs from the original one in that the roles of the objective function and the resource-type constraint are switched, that is, the original operator "min" is replaced by the operator "$\le C$," and the original operator "$\le B$" is replaced by the operator "min." Linear special cases of this problem have been studied by Álvarez-Miranda et al. [1]. Similar to the original problem, we suggest a reduction of the reverse problem to a series of easier ones and, based on it, develop polynomial time algorithms for special cases. These algorithms are asymptotically faster than those in Álvarez-Miranda et al. [1]. Section 5 studies a *bicriteria* version of the problem MIN-RISK, which is denoted as BI-MIN-RISK. It differs from the original problem in that the resource-type constraint of the type "$\le B$" becomes the second objective function to be minimized. For this bicriteria problem, a set of *efficient* (*Pareto optimal*) solutions has to be found such that it contains at least one solution for each point of the *Pareto front* in the corresponding criterion space; see Ehrgott [16] for definitions. Two approaches to solve the bicriteria problem are described. The article concludes with a summary of the obtained results and suggestions for extensions and future research.

2 | REDUCTION TO A SERIES OF EASIER PROBLEMS

We prove properties of optimal solutions of the problem MIN-RISK, and then employ these properties in the reduction of this problem to a series of easier ones. The following theorem is a generalization of Lemma 1 in Chen et al. [9, 10], which, in turn, is a generalization of the result of Dantzig [13], to the case of concave functions f_e.

Theorem 1. *If the problem Min-Risk has a solution, then there exists an optimal solution* (S^*, x^*) *of this problem, in which there is at most one element* $e^* \in S^*$ *such that all variables but variable* $x^*_{e^*}$ *are equal to their lower or upper bounds:* $x^*_e \in \{c_e, d_e\}$, $e \in S^* \setminus \{e^*\}$, *and* $x^*_{e^*} = \min\{d_{e^*}, B - \sum_{e \in S^* \setminus \{e^*\}} x^*_e\}$.

Proof. Consider an optimal solution (S, x), in which there are at least two different elements u and v from S such that $x_u \notin \{c_u, d_u\}$ and $x_v \notin \{c_v, d_v\}$. Without loss of generality, assume that $f_u(x_u+1) - f_u(x_u) \le f_v(x_v) - f_v(x_v - 1)$. For the case of real-valued variables, the unit deviation from x_u and x_v in the above relation should be replaced by an arbitrarily small value $\varepsilon > 0$. Due to the fact that both functions are concave, this relation will remain true if the unit deviation is replaced by any deviation $\delta \ge 1$ (or $\delta \ge \varepsilon$ in the case of the real-valued variables): $f_u(x_u+\delta) - f_u(x_u) \le f_v(x_v) - f_v(x_v - \delta)$. Then, set $\delta = \min\{x_v - c_v, d_u - x_u\}$ and define a new feasible solution (S^0, x^0) such that $S^0 = S$, $x^0_u = x_u + \delta$, $x^0_v = x_v - \delta$ and $x^0_e = x_e$ for all $e \in S \setminus \{u, v\}$. We obtain $\sum_{e \in S^0} x^0_e = \sum_{e \in S} x_e \le B$ and

$$\sum_{e \in S^0} f_e(x^0_e) = \sum_{e \in S \setminus \{u,v\}} f_e(x_e) + f_v(x_v - \delta) + f_u(x_u + \delta)$$
$$= \sum_{e \in S} f_e(x_e) + [f_u(x_u+\delta) - f_u(x_u)] - [f_v(x_v) - f_v(x_v - \delta)] \le \sum_{e \in S} f_e(x_e).$$

Thus, the solution (S^0, x^0) is optimal and the number of variables that are equal to their lower or upper bounds is increased by one. Repetition of this argument at most $|S|-1$ times proves that there exists an optimal solution (S^*, x^*) with at most one variable $x^*_{e^*} \notin \{c_{e^*}, d_{e^*}\}$. For such a solution, we have $c_{e^*} \le x_{e^*} \le \min\{d_{e^*}, B - \sum_{e \in S^* \setminus \{e^*\}} x^*_e\}$. It is clear that resetting $x^*_{e^*} = \min\{d_{e^*}, B - \sum_{e \in S^* \setminus \{e^*\}} x^*_e\}$ does not change the optimality of this solution. ∎

It follows from Theorem 1 that, if the problem MIN-RISK has a solution, then there exists an optimal solution (S^*, x^*) such that $\sum_{e \in S^* \setminus \{e^*\}} f_e(x^*_e) = i^* k_1 + j^* k_2$, where i^* is the number of variables $x^*_e = c_e$ and j^* is the number of variables $x^*_e = d_e$, $e \in S^* \setminus \{e^*\}$. Let K be the set that contains all such pairs (i^*, j^*). It can be easily seen that we can take

$$K = \{(i,j) \mid i + j \le s_{\max} - 1,\ i = 0, 1, \ldots, s_{\max} - 1,\ j = 0, 1, \ldots, s_{\max} - 1\},$$

where $s_{\max}$ is the cardinality of the largest structure in $\mathcal{S}(E)$. The cardinality of the above set K is $|K| = s_{\max}(s_{\max}+1)/2$.

Consider a given element $e_0 \in E$ and a given pair $(i,j) \in K$, which can be viewed as "guesses" such that there exists an optimal solution (S, x) of the problem MIN-RISK, in which $x_e \in \{c_e, d_e\}$ for $e \in S \setminus \{e_0\}$, i is the number of variables $x_e = c_e$ and j is the number of variables $x_e = d_e$, $e \in S \setminus \{e_0\}$. Let us introduce the following auxiliary problem.

Problem $P(e_0, i, j)$.

$$\min_{(S,x)} \sum_{e \in S \setminus \{e_0\}} x_e, \quad \text{subject to}$$
$$e_0 \in S,$$
$$\left|\{e \in S \setminus \{e_0\} \mid x_e = c_e\}\right| = i, \quad \left|\{e \in S \setminus \{e_0\} \mid x_e = d_e\}\right| = j,$$
$$S \in \mathcal{S}(E),$$
$$x_e \in \{c_e, d_e\}, \quad e \in S \setminus \{e_0\}.$$

We denote an optimal solution of the problem $P(e_0, i, j)$ as $(S^{(e_0,i,j)}, x^{(e_0,i,j)})$, if it exists. The following theorem exhibits a relation between an optimal solution of MIN-RISK and that of the auxiliary problem.

Theorem 2. *If the problem MIN-RISK has a solution, then there exists an optimal solution* $(\hat{S}^*, \hat{x}^*)$ *of this problem such that* $\hat{S}^* = S^{(e^*,i^*,j^*)}$, $\hat{x}^*_e = x^{(e^*,i^*,j^*)}_e$ *for* $e \in \hat{S}^* \setminus \{e^*\}$ *and* $\hat{x}^*_{e^*} = \min\{d_{e^*}, B - \sum_{e \in S^{(e^*,i^*,j^*)} \setminus \{e^*\}} x^{(e^*,i^*,j^*)}_e\}$, *for some* $e^* \in E$ *and* $(i^*, j^*) \in K$.

Proof. Consider an optimal solution (S^*, x^*) of the problem MIN-RISK, which satisfies Theorem 1, such that $x^*_e \in \{c_e, d_e\}$ for $e \in S^* \setminus \{e^*\}$, and $x^*_{e^*} = \min\{d_{e^*}, B - \sum_{e \in S^* \setminus \{e^*\}} x^*_e\}$. Due to the feasibility of this solution, we have $c_{e^*} \le x^*_{e^*} \le d_{e^*}$ and $B - \sum_{e \in S^* \setminus \{e^*\}} x^*_e \ge c_{e^*}$.

Let us set

$$i^* = \left|\{e \mid x^*_e = c_e,\ e \in S^* \setminus \{e^*\}\}\right| \quad \text{and} \quad j^* = \left|\{e \mid x^*_e = d_e,\ e \in S^* \setminus \{e^*\}\}\right|.$$

Now, consider the problem $P(e^*, i^*, j^*)$. It has a solution because the solution (S^*, x^*) is feasible for it. Based on an optimal solution $(S^{(e^*,i^*,j^*)}, x^{(e^*,i^*,j^*)})$ of the problem $P(e^*, i^*, j^*)$, construct a solution $(\hat{S}^*, \hat{x}^*)$ such that $\hat{S}^* = S^{(e^*,i^*,j^*)}$, $\hat{x}^*_e = x^{(e^*,i^*,j^*)}_e$ for $e \in S^* \setminus \{e^*\}$ and $\hat{x}^*_{e^*} = \min\{d_{e^*}, B - \sum_{e \in S^{(e^*,i^*,j^*)} \setminus \{e^*\}} x^{(e^*,i^*,j^*)}_e\}$. The latter equality implies $\hat{x}^*_{e^*} \le d_{e^*}$. Let us show that

$\hat{x}^*_{e^*} \ge c_{e^*}$. If $\hat{x}^*_{e^*} = d_{e^*}$, then $\hat{x}^*_{e^*} \ge c_{e^*}$ because $d_{e^*} \ge c_{e^*}$. Otherwise, if $\hat{x}^*_{e^*} = B - \sum_{e\in S^{(e^*,i^*,j^*)}\setminus\{e^*\}} x_e^{(e^*,i^*,j^*)}$, then, due to the relation

$$\sum_{e\in S^{(e^*,i^*,j^*)}\setminus\{e^*\}} x_e^{(e^*,i^*,j^*)} \le \sum_{e\in S^*\setminus\{e^*\}} x^*_e, \tag{1}$$

which follows from the definition of the problem $P(e^*, i^*, j^*)$, we obtain $\hat{x}^*_{e^*} \ge B - \sum_{e\in S^*\setminus\{e^*\}} x^*_e \ge c_{e^*}$. Furthermore,

$$\sum_{e\in\hat{S}^*} \hat{x}^*_e = \sum_{e\in S^{(e^*,i^*,j^*)}\setminus\{e^*\}} x_e^{(e^*,i^*,j^*)} + \min\{d_{e^*}, B - \sum_{e\in S^{(e^*,i^*,j^*)}\setminus\{e^*\}} x_e^{(e^*,i^*,j^*)}\} \le B.$$

We deduce that the solution $(\hat{S}^*, \hat{x}^*)$ is feasible for the problem MIN-RISK. Let us evaluate its objective function value. We have

$$\sum_{e\in\hat{S}^*} f_e(\hat{x}^*_e) = \sum_{e\in S^{(e^*,i^*,j^*)}\setminus\{e^*\}} f_e(x_e^{(e^*,i^*,j^*)}) + f_{e^*}(\hat{x}^*_{e^*}) = i^*k_1 + j^*k_2 + f_{e^*}(\hat{x}^*_{e^*}) = \sum_{e\in S^*\setminus\{e^*\}} f_e(x^*_e) + f_{e^*}(\hat{x}^*_{e^*}). \tag{2}$$

From (1), we obtain

$$\hat{x}^*_{e^*} = \min\left\{d_{e^*}, B - \sum_{e\in S^{(e^*,i^*,j^*)}\setminus\{e^*\}} x_e^{(e^*,i^*,j^*)}\right\} \ge \min\left\{d_{e^*}, B - \sum_{e\in S^*\setminus\{e^*\}} x^*_e\right\} = x^*_{e^*}.$$

Since the function f_{e^*} is nonincreasing, $f_{e^*}(\hat{x}^*_{e^*}) \le f_{e^*}(x^*_{e^*})$. This relation, together with (2), implies $\sum_{e\in\hat{S}^*} f_e(\hat{x}^*_e) \le \sum_{e\in S^*} f_e(x^*_e)$. Thus, the solution $(\hat{S}^*, \hat{x}^*)$ is optimal for the problem MIN-RISK. ■

Based on Theorem 2, the following enumeration procedure can be employed to solve the problem MIN-RISK.

Procedure Enum.

Step 1. Solve $m|K|$ problems $P(e_0, i, j)$, $e_0 \in E$, $(i, j) \in K$. Let Q denote the set of optimal solutions of these problems. Note that it is possible that some problems $P(e_0, i, j)$ do not have a solution.

Step 2. For each solution $(S^{(e_0,i,j)}, x^{(e_0,i,j)}) \in Q$ satisfying $\sum_{e\in S^{(e_0,i,j)}\setminus\{e_0\}} x_e^{(e_0,i,j)} \le B - c_{e_0}$, construct an *extended solution* $(S^{(e_0,i,j)}, \hat{x}^{(e_0,i,j)})$ such that $\hat{x}_e^{(e_0,i,j)} = x_e^{(e_0,i,j)}$ for $e \in S^{(e_0,i,j)} \setminus \{e_0\}$ and $\hat{x}_{e_0}^{(e_0,i,j)} = \min\{d_{e_0}, B - \sum_{e\in S^{(e_0,i,j)}\setminus\{e_0\}} x_e^{(e_0,i,j)}\}$. Let Q_E denote the set of the extended solutions.

Step 3. Select an extended solution $(S^{(e_0,i,j)}, \hat{x}^{(e_0,i,j)}) \in Q_E$ with the minimum value of $ik_1 + jk_2 + f_{e_0}(\hat{x}_{e_0}^{(e_0,i,j)})$. It is optimal for the problem MIN-RISK.

If all the problems $P(e_0, i, j)$, $e_0 \in E$, $(i, j) \in K$, can be solved in $O(T)$ time, then the procedure **Enum** can be implemented to run in $O(T)$ time as well. If all the problems $P(e_0, i, j)$, $(i, j) \in K$, can be solved in $O(D)$ time for any given $e_0 \in E$, then the procedure **Enum** can be implemented to run in $O(mD)$ time. In the next section, we consider special cases of the problem MIN-RISK for which we present polynomial time algorithms for solving all the problems $P(e_0, i, j)$ for all $(i, j) \in K$ and a given $e_0 \in E$, or for all $(i, j) \in K$ and all $e_0 \in E$.

3 | POLYNOMIAL SPECIAL CASES

In this section, we present polynomial time algorithms for several special cases of the problem MIN-RISK.

3.1 | Problem MIN-RISK-SET

In this subsection, we assume that there is a single structure, which is the set E itself, that is, $\mathcal{S}(E) = \{E\}$. The problem MIN-RISK-SET is the following integer program.

$$\min_x \sum_{e=1}^m f_e(x_e), \quad \text{subject to}$$
$$\sum_{e=1}^m x_e \le B,$$
$$x_e \in \{c_e, c_e + 1, \dots, d_e\}, \quad e = 1, \dots, m.$$

In order to facilitate further discussion, let us introduce new variables $\hat{x}_e = x_e - c_e$, functions $\hat{f}_e(\hat{x}_e) = (f_e(\hat{x}_e + c_e) - k_2)/(k_1 - k_2)$, $e \in E$, and $\hat{B} := B - \sum_{e\in E} c_e$. Observe that $\hat{x}_e \in \{0, 1, \dots, d_e - c_e\}$ if $x_e \in \{c_e, c_e + 1, \dots, d_e\}$ and functions $\hat{f}_e(\hat{x}_e)$ are concave nonincreasing, having their values in the interval $[0, 1]$ if the values of $f_e(x_e)$ are such in the interval $[k_2, k_1]$. In other words, it is easy to see that the problem MIN-RISK-SET can be reduced to the same problem with $c_e = 0$, $d_e > 0$, $e \in E$, $0 < B < \sum_{e\in E} d_e$, $k_1 = 1$ and $k_2 = 0$. If $B = 0$, then the solution $x_e = 0$, $e \in E$, is optimal and, if $B \geq \sum_{e\in E} d_e$, then the solution $x_e = d_e$, $e \in E$, is optimal.

Comment 2. The above reduction is only valid for the case of a single structure $\mathcal{S}(E) = \{E\}$ and it is not valid if $\mathcal{S}(E) \neq \{E\}$, because then the modified upper bound $\hat{B}$ would be structure dependent.

We denote the reduced problem as REDUCED-MIN-RISK-SET. It can be constructed in $O(m)$ time. We now describe algorithms with running times $O(m^2)$, $O(m \log m)$ and $O(m)$ for the problem REDUCED-MIN-RISK-SET. For this problem, $|S| = m$ for the unique $S = E$. Therefore, $K = \{(m-1-j, j) \mid j = 0, 1, \ldots, m-1\}$. Renumber elements of the set $E \setminus \{e_0\}$ such that $E \setminus \{e_0\} = \{1, \ldots, m-1\}$. Problem $P(e_0, m-1-j, j)$, $e_0 \in E$, $(m-1-j, j) \in K$, related to the problem REDUCED-MIN-RISK-SET, can be formulated as follows.

$$\min_x \sum_{e=1}^{m-1} x_e, \quad \text{subject to}$$
$$|\{e | 1 \leq e \leq m-1, x_e = d_e\}| = j,$$
$$x_e \in \{0, d_e\}, \quad e = 1, \ldots, m-1.$$

Introduce 0–1 variables y_e such that $y_e = 0$ if $x_e = 0$ and $y_e = 1$ if $x_e = d_e$, $e = 1, \ldots, m-1$. Denote $y = (y_1, \ldots, y_{m-1})$. Problem $P(e_0, m-1-j, j)$ reduces to the following well-known SELECTION problem (see, e.g., Lachmann et al. [30]).

$$\min_y \sum_{e=1}^{m-1} d_e y_e, \quad \text{subject to}$$
$$\sum_{e=1}^{m-1} y_e = j,$$
$$y_e \in \{0, 1\}, \quad e = 1, \ldots, m-1.$$

This problem is solved by setting $y_e = 1$ for the j smallest values d_e. Elements of the set E can be renumbered in the nondecreasing order of d_e in $O(m \log m)$ time. After that, all the problems $P(e_0, m-1-j, j)$, $j = 0, 1, \ldots, m-1$, for a given $e_0 \in E$ can be solved in $O(m)$ time. Thus, the problem REDUCED-MIN-RISK-SET can be solved in $O(m^2)$ time by employing the procedure **Enum**. Below we suggest algorithms with better asymptotic running times for this problem.

Let us make several useful observations. Assume that $d_1 \leq \cdots \leq d_m$. Define $D_0 = 0$ and $D_j = \sum_{e=1}^{j} d_e$ for $j = 1, \ldots, m$. Since $d_e > 0$ for any $e \in E$, there exists an index j^*, $0 \leq j^* \leq m-1$, such that

$$D_0 < D_1 < \cdots < D_{j^*} \leq B < D_{j^*+1} < \cdots < D_m. \tag{3}$$

Now, define $D_{e_0,0} = 0$, $D_{e_0,j} = D_j$ for $j+1 \leq e_0 \leq m$ and $D_{e_0,j} = D_{j+1} - d_{e_0}$ for $1 \leq e_0 \leq j$, $j = 1, \ldots, m-1$. Observe that $D_j \leq D_{e_0,j} < D_{j+1}$ for any $e_0 \in E$ and $j = 0, 1, \ldots, m-1$, because if $e_0 \geq j+1$, then $D_{e_0,j} = D_j$, and if $e_0 \leq j$, then $D_{e_0,j} = D_{j+1} - d_{e_0} = D_j + d_{j+1} - d_{e_0}$, where $0 \leq d_{j+1} - d_{e_0} < d_{j+1}$. Therefore, for any $e_0 \in E$, either

$$D_{e_0,j^*-1} < D_{j^*} \leq B < D_{e_0,j^*} < D_{j^*+1} \leq D_{e_0,j^*+1}, \tag{4}$$

or

$$D_{j^*} \leq D_{e_0,j^*} \leq B < D_{j^*+1} \leq D_{e_0,j^*+1}. \tag{5}$$

Thus, if the B value breaks the sequence $D_0, D_1, \ldots, D_m$ between D_{j^*} and D_{j^*+1}, then it breaks the sequence $D_{e_0,0}, D_{e_0,1}, \ldots, D_{e_0,m}$ either between D_{e_0,j^*-1} and D_{e_0,j^*} or between D_{e_0,j^*} and D_{e_0,j^*+1} for any $e_0 \in E$.

Denote by $F^*_{e_0,j}$ the value of a potential optimal solution for the problem REDUCED-MIN-RISK-SET, in which $x_e = d_e$ for j smallest values d_e, $e \in E \setminus \{e_0\}$, $0 \leq x_{e_0} \leq d_{e_0}$ for a given $e_0 \in E$, $j = 0, 1, \ldots, m-1$, and the remaining variables are equal to zero. Due to Theorem 1,

$$F^*_{e_0,j} = (m-1-j) + f_{e_0}(\min\{d_{e_0}, B - D_{e_0,j}\}) \tag{6}$$

if such a solution exists, that is, if $D_{e_0,j} \leq B$, for $e_0 \in E$, $j = 0, 1, \ldots, m-1$. Define $F^*_{e_0,j} = \infty$ if no such solution exists, that is, if $D_{e_0,j} > B$. Observe that, if $D_{e_0,j} \leq B$, then $0 \leq f_{e_0}(\min\{d_{e_0}, B - D_{e_0,j}\}) \leq 1$. This observation, together with (4), (5), and (6) implies $F^*_{e_0,0} \geq F^*_{e_0,1} \geq \cdots \geq F^*_{e_0,j^*-1} \geq \min\{F^*_{e_0,j^*}, F^*_{j^*+1,j^*}\}$, $F^*_{j^*+1,j^*} < \infty$ and $F^*_{e_0,j^*+k} = \infty$, $k = 1, \ldots, m-j^*$. For a given $e_0 \in E$, if (4) is satisfied, then $F^*_{e_0,j^*} = \infty$ and a solution corresponding to $F^*_{j^*+1,j^*}$ is the best, and if (5) is satisfied, then a solution corresponding to $F^*_{e_0,j^*}$ is the best. We deduce that an optimal solution corresponds to the value $F^*_{e^*,j^*} = \min\{F^*_{e_0,j^*} | e_0 \in E\}$.

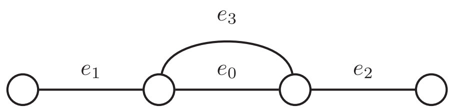

FIGURE 1 A subgraph of G containing e_0

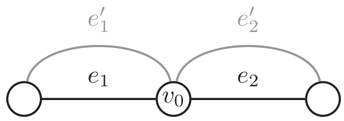

FIGURE 2 A subgraph of G^* containing v_0

Since values $D_1, \ldots, D_m$ satisfying (3) can be calculated in $O(m)$ time, provided that the upper bounds d_e are sorted in the nondecreasing order, index j^* can be found in $O(m)$ time by scanning values $D_1, \ldots, D_m$, or in $O(\log m)$ time by a bisection search. After that, values $F^*_{e_0,j^*}$ for all $e_0 \in E$ can be calculated in $O(m)$ time by (6). An optimal solution corresponding to $F^*_{e^*,j^*}$ can be determined as follows. If $j^*+1 \le e^* \le m$, then $x^*_e = d_e$ for $e = 1, \ldots, j^*$, $x^*_e = 0$ for $e = j^*+1, \ldots, m$ and $e \ne e^*$, and $x^*_{e^*} = B - D_{j^*}$. If $1 \le e^* \le j^*$, then $x^*_e = d_e$ for $e = 1, \ldots, j^*+1$ and $e \ne e^*$, $x^*_e = 0$ for $e = j^*+2, \ldots, m$, and $x^*_{e^*} = B - D_{j^*+1} + d_{e^*}$.

We have shown that the problem Reduced-Min-Risk-Set can be solved in $O(m \log m)$ time. Let us now show that it can be solved in $O(m)$ time. Assume that the upper bounds d_e, $e \in E$, are not sorted. Consider a permutation $(i_1, \ldots, i_m)$ such that $d_{i_1} \le \cdots \le d_{i_m}$ and assume that j^*, i_{j^*} and i_{j^*+1} are given such that $\sum_{j=0}^{j^*} d_{i_j} \le B < \sum_{j=0}^{j^*+1} d_{i_j}$, $0 \le j^* \le m-1$, where $d_{i_0} = 0$. Then, calculate in $O(m)$ time the set $X^{(1)}_{j^*} = \{j | d_j < d_{i_{j^*}}, j = 1, \ldots, m\}$ and the set $X^{(2)}_{j^*} = \{j | d_j = d_{i_{j^*}}, j = 1, \ldots, m\}$ of cardinality $|X^{(2)}_{j^*}| = j^* - |X^{(1)}_{j^*}|$. Calculate the set $X_{j^*} = X^{(1)}_{j^*} \cup X^{(2)}_{j^*}$ of cardinality j^*. We have $D_{j^*} = \sum_{e \in X_{j^*}} d_e$, $D_{e_0,j^*} = D_{j^*}$ if $e_0 \notin X_{j^*}$, $D_{e_0,j^*} = D_{j^*} + d_{i_{j^*+1}} - d_{e_0}$ if $e_0 \in X_{j^*}$, and $F^*_{e_0,j^*} = (m-1-j^*) + f_{e_0}(\min\{d_{e_0}, B - D_{e_0,j^*}\})$. Thus, for given j^*, i_{j^*} and i_{j^*+1}, all the values $F^*_{e_0,j^*}$, and hence, the optimal solution value $F^*_{e^*,j^*}$, can be calculated in $O(m)$ time. The corresponding optimal solution can be determined as follows. If $e^* \notin X_{j^*}$, then $x^*_e = d_e$ for $e \in X_{j^*}$, $x^*_e = 0$ for $e \notin X_{j^*}$ and $e \ne e^*$, and $x^*_{e^*} = B - D_{j^*}$. If $e^* \in X_{j^*}$, then $x^*_e = d_e$ for $e \in (X_{j^*} \cup \{i_{j^*+1}\}) \setminus \{e^*\}$, $x^*_e = 0$ for $e \notin (X_{j^*} \cup \{i_{j^*+1}\}) \setminus \{e^*\}$, and $x^*_{e^*} = B - D_{j^*+1} + d_{e^*}$.

It remains to show that j^*, i_{j^*} and i_{j^*+1} can be found in $O(m)$ time. The indices j^* and i_{j^*} can be found in $O(m)$ time by the weighted median technique described, for example, by Korte and Vygen [26, p. 459] for the fractional knapsack problem. Initialize the set of indices $\{i_1, \ldots, i_{j^*}\}$ as $Y = \emptyset$. The idea is to find in $O(m)$ time the median of the numbers d_e, $e \in E$. This can be done by the algorithm of Blum et al. [4]. Then, again in $O(m)$ time, determine the set $X^{(1)}$ of items, whose values d_e are strictly less than the median, the set $X^{(2)}$ of items, whose values d_e are equal to the median and whose cardinality is $|X^{(2)}| = \lceil m/2 \rceil - |X^{(1)}|$. Calculate the set $X = X^{(1)} \cup X^{(2)}$ of cardinality $\lceil m/2 \rceil$, and the value $D(X) = \sum_{e \in X} d_e$. If $D(X) \le B$, then reset $Y = Y \cup X$, $E = E \setminus X$, $m = m - \lceil m/2 \rceil$, $B = B - D(X)$, and recursively solve the median finding problem. If $D(X) > B$, then reset $E = X$, $m = \lceil m/2 \rceil$, and recursively solve the median finding problem. The procedure stops when $E = \emptyset$. The index j^* is equal to $|Y|$ and the index i_{j^*} is equal to the index of the largest d_e, $e \in Y$. The required time is determined by the formula $T(m) = T(m/2) + O(m)$, which gives $T(m) = O(m)$. Having j^* and i_{j^*} detected, the index i_{j^*+1} can be found in $O(m)$ time by determining the minimal value d_e for $e \in E \setminus Y$.

Corollary 1. *The problem Min-Risk-Set can be solved in $O(m)$ time.*

3.2 | Problem Min-Risk-Span

In the problem Min-Risk-Span, the elements are the edges and the structures are the *spanning trees* of an undirected graph $G = (V, E)$, $|V| = n$, $|E| = m$. Recall that a spanning tree $T = (V, E_T)$, $E_T \subseteq E$, is a subgraph of G that includes all nodes of G, and it is a tree. By the definition of the spanning tree, $|E_T| = n-1$. Therefore, for the problem Min-Risk-Span, $K = \{(i, n-2-i) \mid i = 0, 1, \ldots, n-2\}$ and $|K| = n-1$.

For given $e_0 \in E$ and $(i, n-2-i) \in K$, each problem $P(e_0, i, n-2-i)$ can be solved by the following approach. First, construct a *graph* $G^* = (V^*, E^*)$, which differs from the original graph $G = (V, E)$ in that there is one extra copy of each edge from E but the edge e_0 and the edges parallel to e_0 are *contracted* to a single node v_0 so that the other edges incident to the ends of e_0 in G become incident to v_0 in G^*; see Figures 1 and 2 for an illustration.

Consider a pair of edges e and e', where e is an original edge from $E \setminus \{e_0\}$ and e' is its copy in E^*. We call the edge e *red* and associate weight $w_e = c_e$ with it. We call the edge e' *white* and associate weight $w_{e'} = d_e$ with it.

Theorem 3. *The problem $P(e_0, i, n-2-i)$ on the graph G reduces to the problem of finding a minimum weight spanning tree with exactly i red edges on the graph G^*, which we denote as Span-i-red.*

Proof. Let $(S^{(e_0,i)}, x^{(e_0,i)})$ be a solution of the problem $P(e_0, i, n-2-i)$. By the definition, $S^{(e_0,i)}$ is a spanning tree in G, which necessarily includes edge e_0. Transform $S^{(e_0,i)}$ into a solution T of the corresponding problem SPAN-i-RED as follows. Contract the edge e_0 into the node v_0. For $e \in S^{(e_0,i)} \setminus \{e_0\}$, if $x_e^{(e_0,i)} = c_e$, then include the red edge e and, if $x_e^{(e_0,i)} = d_e$, then include the white edge e' into T. By the definition of the solution $S^{(e_0,i)}$, the number of variables $x_e^{(e_0,i)} = c_e$ is equal to i. Since the edge contraction does not create any cycle in the tree, we deduce that T is a spanning tree with exactly i red edges in G^*. Its total weight is equal to the optimal objective value of the problem $P(e_0, i, n-2-i)$.

Now, let T be a solution of the problem SPAN-i-RED. It necessarily includes node v_0. Based on T, construct a solution $(S^{(e_0,i)}, x^{(e_0,i)})$ of the corresponding problem $P(e_0, i, n-2-i)$, in which, if one of the coupled edges e and e' is included into the solution T, then $e \in S^{(e_0,i)}$. Furthermore, if the red edge e is included, then $x_e^{(e_0,i)} = c_e$, and, if the white edge e' is included, then $x_e^{(e_0,i)} = d_e$. Therefore, $|\{e \mid x_e^{(e_0,i)} = c_e\}| = i$. Replace the node v_0 in T by the edge e_0 in $S^{(e_0,i)}$. Since *cleaving* a node in a tree does not create any cycle, structure $S^{(e_0,i)}$ is a spanning tree for the graph G. The objective value of $P(e_0, i, n-2-i)$ on $S^{(e_0,i)}$ is equal to the optimal objective value of the problem SPAN-i-RED. ■

Gabow and Tarjan [17, Theorem 4.1, p. 97] developed an $O(tree(m,n) + n\log n)$ time algorithm to solve the problem SPAN-i-RED, where $tree(m,n)$ is the time of solving the classical minimum weight spanning tree problem in a graph with m edges and n nodes. If the algorithm of Chazelle [8] is used, then $tree(m,n) = O(m\alpha(m,n))$, where $\alpha(\cdot,\cdot)$ is the inverse Ackermann function. By Theorem 3, the approach of Gabow and Tarjan [17] can be employed to solve all the problems $P(e_0, i, n-2-i)$, $(i, n-2-i) \in K$, $e_0 \in E$, in $O(mn(m\alpha(m,n) + n\log n))$ time. Hence, the same time is needed to solve the problem MIN-RISK-SPAN by the procedure **Enum**. Chen et al. [10] proposed an $O(m^2 \log m \log n(m + n\log n))$ time algorithm for the special case of the problem MIN-RISK-SPAN in which the risk functions are $f_e(x_e) = \frac{d_e - x_e}{d_e - c_e}$.

Corollary 2. *The problem Min-Risk-Span can be solved in* $O(mn(m\alpha(m,n) + n\log n))$ *time.*

Comment 3. If all the values $c_e, e \in E$, are distinct and all the values $d_e, e \in E$, are distinct, then the algorithm of Gusfield [19] can be used to solve all the problems SPAN-i-RED, $i = 0, 1, \dots, n-2$, in $O(n^2)$ time. In this case, the procedure **Enum** will run in $O(mn^2)$ time.

3.3 | Problem MIN-RISK-PATH

In the problem MIN-RISK-PATH, the elements are the arcs and the structures are the paths between two specified nodes a and b of a directed acyclic graph $G = (V, E)$, $|V| = n$, $|E| = m$. Since each path of an acyclic graph includes at most n nodes and, therefore, at most $n-1$ arcs, for the problem MIN-RISK-PATH we have $s_{\max} \le n-1$, and we can set $K = \{(i,j) \mid i = 0, 1, \dots, n-2, j = 0, 1, \dots, n-2, \ i + j \le n-2\}$. Hence, $|K| = (n-1)n/2$.

Comment 4. If G would contain cycles, then the problem MIN-RISK-PATH would be strongly NP-hard because the strongly NP-complete problem HAMILTONIAN PATH reduces to it by setting $c_e = 0$, $d_e = 1$, $f_e(0) = 0$, $f_e(1) = -1$, $e \in E$, and $B = n-1$. If the optimal value of this problem is equal to $1-n$, then the graph contains a Hamiltonian path, otherwise, it contains no Hamiltonian path.

For given $e_0 \in E$ and $(i,j) \in K$, each problem $P(e_0, i, j)$ can be solved by the following approach. Construct an acyclic graph $G^* = (V, E^*)$, which differs from the original graph $G = (V, E)$ in that there is an extra copy, e', of each arc $e \in E \setminus \{e_0\}$. Each arc $u \in E^*$ is associated with a weight w_u and values of three resources $r_u^{(1)}$, $r_u^{(2)}$, and $r_u^{(3)}$. For the coupled arcs e and e', we define $w_e = c_e$, $r_e^{(1)} = 1$, $r_e^{(2)} = 0$, $r_e^{(3)} = 0$, and $w_{e'} = d_e$, $r_{e'}^{(1)} = 0$, $r_{e'}^{(2)} = 1$, $r_{e'}^{(3)} = 0$. For the arc e_0, we define $w_{e_0} = 0$, $r_{e_0}^{(1)} = 0$, $r_{e_0}^{(2)} = 0$, $r_{e_0}^{(3)} = 1$.

Consider the problem of finding a minimum weight path S in the graph G^* between nodes a and b such that the total values of the resources 1, 2, and 3 along this path are equal to i, j, and 1, respectively: $\sum_{e\in S} r_e^{(1)} = i$, $\sum_{e\in S} r_e^{(2)} = j$, and $\sum_{e\in S} r_e^{(3)} = 1$. We denote this problem as RESOURCE-PATH(i, j, e_0).

Theorem 4. *The problem* $P(e_0, i, j)$ *reduces to the problem Resource-Path*(i, j, e_0).

Proof. Let $(S^{(e_0,i,j)}, x^{(e_0,i,j)})$ be a solution of the problem $P(e_0, i, j)$. By definition, $S^{(e_0,i,j)}$ is a path in G, which necessarily goes via the arc e_0. For the problem RESOURCE-PATH(i, j, e_0), construct a path S, in which $e \in S^{(e_0,i,j)}$ implies that one of the coupled arcs e and e' is included into the path S. If $x_e^{(e_0,i,j)} = c_e$, then the arc e is included and, if $x_e^{(e_0,i,j)} = d_e$, then the arc e' is included. By the definition of the solution $S^{(e_0,i,j)}$, the number of variables $x_e^{(e_0,i,j)} = c_e$ is equal to i and the number of variables $x_e^{(e_0,i,j)} = d_e$ is equal to j. We deduce that the path S is a solution of the problem of finding a minimum weight path in the graph G^* between the nodes a and b such that the total values of the resources 1, 2, and 3 along this path are equal to i, j, and 1, respectively: $\sum_{e \in S} r_e^{(1)} = i$, $\sum_{e \in S} r_e^{(2)} = j$, and $\sum_{e \in S} r_e^{(3)} = 1$. Its weight is equal to the optimal objective value of the problem $P(e_0, i, j)$.

Now, let a path S be a solution of the problem RESOURCE-PATH(i, j, e_0). It obligatorily goes via the arc e_0 as it is the only arc with the nonzero value $r_{e_0}^{(3)} = 1$ of the third resource. Construct a solution $(S^{(e_0,i,j)}, x^{(e_0,i,j)})$ of the corresponding problem $P(e_0, i, j)$, in which if one of the coupled arcs e and e' is included into the path, then $e \in S^{(e_0,i,j)}$. Furthermore, if the arc e is included, then $x_e^{(e_0,i,j)} = c_e$, and, if the arc e' is included, then $x_e^{(e_0,i,j)} = d_e$. Therefore, $|\{e | x_e^{(e_0,i,j)} = c_e\}| = i$ and $|\{e | x_e^{(e_0,i,j)} = d_e\}| = j$. The objective value of the solution $(S^{(e_0,i,j)}, x^{(e_0,i,j)})$ is equal to the optimal objective value of the problem RESOURCE-PATH(i, j, e_0). ■

All the problems RESOURCE-PATH(i, j, e_0), $e_0 \in E$, $(i, j) \in K$, $i = 0, 1, \dots, n-2$, $j = 0, 1, \dots, n-2$, can be solved in $O(mn^2)$ time by the dynamic programming algorithm of Desrochers [14], which is an extension of the algorithm of Joksch [24] for the problem with one resource. The running time of the algorithm in Desrochers [14] is $O(mR_1R_2R_3)$, where R_h is an upper bound on the total value of the resource h, $h = 1, 2, 3$. Hence, the problem MIN-RISK-PATH can be solved in $O(mn^2)$ time by employing the procedure **Enum**.

Corollary 3. *The problem Min-Risk-Path can be solved in $O(mn^2)$ time.*

Comment 5. If $k_1 = 0$ (or $k_2 = 0$) in Assumption 1, then there is no need to enumerate values i (resp., values j) in the pairs $(i, j) \in K$ because they have no effect on the objective function value. Therefore, if $k_1 = 0$ or $k_2 = 0$, then the problem MIN-RISK-PATH can be solved in $O(mn)$ time.

Chen et al. [9] presented an $O(mn^3)$ time algorithm to solve the case of the problem MIN-RISK-PATH, in which the risk functions are $f_e(x_e) = \frac{d_e - x_e}{d_e - c_e}$, and hence, $k_2 = 0$.

4 | REVERSE PROBLEMS

In this section, a reverse version of the problem MIN-RISK is studied. Let C be a given real number. The reverse problem can be formulated as follows.

Problem REV-MIN-RISK.

$$\min_{(S,x)} \sum_{e \in S} x_e, \quad \text{subject to}$$
$$\sum_{e \in S} f_e(x_e) \le C,$$
$$S \in \mathcal{S}(E),$$
$$x_e \in \{c_e, c_e + 1, \dots, d_e\}, \quad e \in S.$$

The problem REV-MIN-RISK is a generalization of the risk constrained problem introduced by Álvarez-Miranda et al. [1], which we denote as REV-MIN-LINEAR-RISK.

Problem REV-MIN-LINEAR-RISK.

$$\min_{(S,x)} \sum_{e \in S} x_e, \quad \text{subject to}$$
$$\sum_{e \in S} \frac{d_e - x_e}{d_e - c_e} \le C,$$
$$S \in \mathcal{S}(E),$$
$$x_e \in \{c_e, c_e + 1, \dots, d_e\}, \quad e \in S.$$

Let us prove properties of an optimal solution of the problem REV-MIN-RISK similar to those for the problem MIN-RISK. The following theorem is a generalization of Lemma 2.2 in Álvarez-Miranda et al. [1] to the case of the concave functions f_e.

Theorem 5. *If the problem Rev-Min-Risk has a solution, then there exists an optimal solution (S^*, x^*) of this problem, in which there is at most one element $e^* \in S^*$ such that all variables but variable $x_{e^*}^*$ are equal to their lower or upper bounds: $x_e^* \in \{c_e, d_e\}$, $e \in S^* \setminus \{e^*\}$, and $x_{e^*}^* = \min\{x | x \in \{c_{e^*}, c_{e^*} + 1, \dots, d_{e^*}\}, f_{e^*}(x) \le C - \sum_{e \in S^* \setminus \{e^*\}} f_e(x_e^*)\}$.*

Proof. Let (S^0, x^0) be an optimal solution of REV-MIN-RISK and F^0 be the value of its objective function, that is, $F^0 = \sum_{e \in S^0} x_e^0$. Now, consider the problem MIN-RISK, in which F^0 is the right-hand side of its constraint. It is easy to see that (S^0, x^0) is a feasible solution for the latter problem. By Theorem 1, the problem MIN-RISK has an optimal solution (S^*, x^*) such that $x_e^* \in \{c_e, d_e\}$, $e \in S^* \setminus \{e^*\}$, and $x_{e^*}^* = \min\{d_{e^*}, F^0 - \sum_{e \in S^* \setminus \{e^*\}} x_e^*\}$. From the optimality of (S^*, x^*) for MIN-RISK and the feasibility of (S^0, x^0) for both MIN-RISK and REV-MIN-RISK, we deduce that $\sum_{e \in S^*} x_e^* \le \sum_{e \in S^0} x_e^0 = F^0$ and $\sum_{e \in S^*} f_e(x_e^*) \le \sum_{e \in S^0} f_e(x_e^0) \le C$, which implies that (S^*, x^*) is optimal for the problem REV-MIN-RISK. It is clear that resetting $x_{e^*}^* = \min\{x | x \in \{c_{e^*}, c_{e^*} + 1, \ldots, d_{e^*}\}, f_{e^*}(x) \le C - \sum_{e \in S^* \setminus \{e^*\}} f_e(x_e^*)\}$ does not change the optimality of this solution. ∎

Based on Theorem 5, enumeration procedure similar to **Enum** can be employed to solve the problem REV-MIN-RISK.
Procedure Rev-Enum.

Step 1. Solve $m|K|$ problems $P(e_0, i, j)$, $e_0 \in E$, $(i, j) \in K$. Let Q denote the set of optimal solutions of these problems.

Step 2. For each solution $(S^{(e_0,i,j)}, x^{(e_0,i,j)}) \in Q$ such that $\sum_{e \in S^{(e_0,i,j)} \setminus \{e_0\}} f_e(x_e^{(e_0,i,j)}) \le C - f_{e_0}(d_{e_0})$, construct an *extended solution* $(S^{(e_0,i,j)}, \hat{x}^{(e_0,i,j)})$ such that $\hat{x}_e^{(e_0,i,j)} = x_e^{(e_0,i,j)}$ for $e \in S^{(e_0,i,j)} \setminus \{e_0\}$ and $\hat{x}_{e_0}^{(e_0,i,j)} = \min\{x | x \in \{c_{e_0}, c_{e_0} + 1, \ldots, d_{e_0}\}, f_{e_0}(x) \le C - \sum_{e \in S^{(e_0,i,j)} \setminus \{e_0\}} f_e(x_e^{(e_0,i,j)})\}$. Let Q_E denote the set of the extended solutions.

Step 3. Select an extended solution $(S^{(e_0,i,j)}, \hat{x}^{(e_0,i,j)}) \in Q_E$ with the minimum value of $\sum_{e \in S^{(e_0,i,j)}} \hat{x}_e^{(e_0,i,j)}$. It is optimal for the problem REV-MIN-RISK.

The procedure **Rev-Enum** can be implemented to run in $O(R_1 + m|K|R_2)$ time, where R_1 is the time complexity of its Step 1 and R_2 is the time complexity of computing $\min\{x | x \in \{c_{e_0}, c_{e_0}+1, \ldots, d_{e_0}\}, f_{e_0}(x) \le C - \sum_{e \in S^{(e_0,i,j)} \setminus \{e_0\}} f_e(x_e^{(e_0,i,j)})\}$ for any $e_0 \in E$ in its Step 2. Since all the functions f_e are nonincreasing, the latter minimum can be calculated by a bisection search in $O(\log W)$ time, where $W = \max_{e \in E}\{d_e - c_e + 1\}$. Thus, the procedure **Rev-Enum** can be implemented to run in $O(R_1 + m|K| \log W)$ time.

For the minimum spanning tree and shortest path cases of the problem REV-MIN-RISK, algorithms in Sections 3.2 and 3.3, respectively, can be used in Step 1 of the procedure **Rev-Enum**. The set case of this problem is the following integer program.

Problem REV-MIN-RISK-SET.

$$\min_x \sum_{e \in E} x_e, \quad \text{subject to}$$
$$\sum_{e \in E} f_e(x_e) \le C,$$
$$x_e \in \{c_e, c_e + 1, \ldots, d_e\}, \quad e \in E.$$

This problem can be solved in $O(m^2 \log W)$ time by applying the procedure **Rev-Enum**. Below, we describe a linear time algorithm for it. As for the problem MIN-RISK-SET in Section 3.1, by introducing new variables $\hat{x}_e = x_e - c_e$, functions $\hat{f}_e(\hat{x}_e) = (f_e(\hat{x}_e + c_e) - k_2)/(k_1 - k_2)$, $e \in E$, and $\hat{C} = (C - mk_2)/(k_1 - k_2)$, the problem REV-MIN-RISK-SET reduces to the same problem with $c_e = 0$, $d_e > 0$, $e \in E$, $0 < C < m$, $k_1 = 1$ and $k_2 = 0$. If $C = 0$, then, for the new problem, the solution $x_e = d_e$ with $f_e(x_e) = 0$, $e \in E$, is optimal. If $C \ge m$, then the solution $x_e = 0$ with $f_e(x_e) = 1$, $e \in E$, is optimal. We denote this reduced problem as REDUCED-REV-MIN-RISK-SET. It can be constructed in $O(m)$ time. Denote $H(x) = \sum_{e \in E} x_e$.

Following Theorem 5, there exists $e_0 \in E$ such that the problem REDUCED-REV-MIN-RISK-SET has an optimal solution, denoted by $x^{e_0 j}$, in which $x_e^{e_0 j} = d_e$ for j smallest values d_e, $e \in E \setminus \{e_0\}$, $x_e^{e_0 j} = 0$ for the remaining $e \in E$,

$$x_{e_0}^{e_0 j} = \min\{x | 0 \le x \le d_{e_0}, f_{e_0}(x) \le C - (m - 1 - j)\} \tag{7}$$

and $H(x^{e_0 j}) = D_{e_0 j} + x_{e_0}^{e_0 j}$, where $D_{e_0 j}$ is defined in Section 3.1. From (7), we deduce that $j \ge \lceil m - 1 - C + f_{e_0}(x_{e_0}^{e_0 j}) \rceil \ge j_0 = \lceil m - 1 - C \rceil$. As a consequence, the problem REDUCED-REV-MIN-RISK-SET has a solution for $j = j_0, j_0 + 1, \ldots, m - 1$, and it does not have a solution for $j = 0, 1, \ldots, j_0 - 1$, where $0 \le j_0 \le m - 1$.

Observe that $f_{e_0}(x_{e_0}^{e_0 j}) \le C - (m - 1 - j_0)$ implies $C - (m - 1 - j_0) \ge 0$ and $f_{e_0}(0) = 1 \le C - (m - 1 - j)$ for any $j \ge j_0 + 1$, that is, $x_{e_0}^{e_0 j} = 0$ for $j \ge j_0 + 1$. Therefore, $H(x^{e_0 j}) = D_{e_0 j}$ for $j \ge j_0 + 1$. Since values $D_{e_0 j}$ are nondecreasing (see Section 3.1), we deduce that $H(x^{e_0 j_0 + 1}) \le H(x^{e_0 j_0 + k})$, $k = 2, 3, \ldots, m - 1 - j_0$. Thus, an optimal solution of the problem REDUCED-REV-MIN-RISK-SET is $x^{e_0 j_0}$ or $x^{e_0 j_0 + 1}$ for some $e_0 \in E$.

Let $x^{e^* j^*}$ be an optimal solution. Its value can be calculated as $H(x^{e^* j^*}) = \min_{e_0 \in E}\{D_{e_0 j_0} + x_{e_0}^{e_0 j_0}, D_{e_0 j_0 + 1}\}$. Similar to Section 3.1, all the values $D_{e_0 j_0}$ and $D_{e_0 j_0 + 1}$, $e_0 \in E$, can be calculated in $O(m)$ time by employing the weighted median technique to determine j_0 smallest numbers d_e, $e \in E$. If functions f_e are arbitrary concave nonincreasing, then the value $x_{e_0}^{e_0 j_0}$ can be calculated in $O(\log d_{e_0})$ time by a bisection search, hence, all these values can be calculated in $O(\sum_{e \in E} \log d_e)$ time. Therefore, the problem REDUCED-REV-MIN-RISK-SET can be solved in $O(m + \sum_{e \in E} \log d_e)$ time.

Corollary 4. *The problem Rev-Min-Risk-Set can be solved in* $O(m + \sum_{e\in E} \log d_e)$ *time.*

Comment 6. If the minimum in (7) or the inverse functions f_e^{-1}, $e \in E$, can be computed in $O(1)$ time, then the problem REV-MIN-RISK-SET can be solved in $O(m)$ time.

5 | BICRITERIA PROBLEMS

In the problem BI-MIN-RISK, the functions $F_1(S,x) = \sum_{e\in S} f_e(x_e)$ and $F_2(S,x) = \sum_{e\in S} x_e$ have to be minimized on the set of solutions (S,x) such that $S \in \mathcal{S}(E)$ and $x_e \in \{c_e, c_e+1, \ldots, d_e\}$, $e \in S$.

The procedure **Enum** can be modified to construct a set of efficient solutions of the bicriteria problem BI-MIN-RISK. Below we give two variants of such a modified procedure. The first variant is an application of the well-known *ε-constraint method* (see Cheng et al. [11], Ehrgott [16], and T'kindt and Billaut [35]), while the second variant constructs feasible solutions for a set of points that is broader than the Pareto front and then applies the algorithm of Kung et al. [27] to select efficient solutions from it. Let B^0 be an upper bound on the values $F_2(S,x)$. For example, we can set $B^0 = s_{\max} \cdot d_{\max}$, where $d_{\max} = \max_{e\in E}\{d_e\}$.

Procedure Bi-Enum-1.

Step 1. Solve $m|K|$ problems $P(e_0,i,j)$, $e_0 \in E$, $(i,j) \in K$. Let Q denote the set of optimal solutions of these problems.

Step 2. For each value $B \in \{1,2,\ldots,B^0\}$, find a *B-best solution* $(S^{(e_0,i,j)}, x^{(e_0,i,j)}) \in Q$ such that $\sum_{e\in S^{(e_0,i,j)}\setminus\{e_0\}} x_e^{(e_0,i,j)} \le B - c_{e_0}$, the vector $(ik_1 + jk_2 + f_{e_0}(y_{e_0}), \sum_{e\in S^{(e_0,i,j)}\setminus\{e_0\}} x_e^{(e_0,i,j)} + y_{e_0})$ is minimum with respect to the first coordinate and, if there are several vectors with the first coordinate being minimum, it is minimum with respect to the second coordinate, where $y_{e_0} = \min\{d_{e_0}, B - \sum_{e\in S^{(e_0,i,j)}\setminus\{e_0\}} x_e^{(e_0,i,j)}\}$. For each B-best solution $(S^{(e_0,i,j)}, x^{(e_0,i,j)})$, construct an extended B-best solution $(S^{(B)}, x^{(B)})$ such that $x_e^{(B)} = x_e^{(e_0,i,j)}$ for $e \in S^{(e_0,i,j)} \setminus \{e_0\}$ and $x_{e_0}^{(B)} = y_{e_0}$. The point $(F_1(S^{(B)}, x^{(B)}), F_2(S^{(B)}, x^{(B)}))$ belongs to the Pareto front P. Add it to the set P.

Step 3. Output the set of the extended B-best solutions and the Pareto front P for the problem BI-MIN-RISK.

The procedure **Bi-Enum-1** can be implemented to run in $O(R_1' + m|K|B^0)$ time, where R_1' is the time complexity of its Step 1. Justification of the correctness of procedure **Bi-Enum-1** is similar to that for the general ε-constraint method.

Procedure Bi-Enum-2.

Step 1. Solve $m|K|$ problems $P(e_0,i,j)$, $e_0 \in E$, $(i,j) \in K$. Let Q denote the set of optimal solutions of these problems.

Step 2. For each solution $(S^{(e_0,i,j)}, x^{(e_0,i,j)}) \in Q$, construct $d_{e_0} - c_{e_0} + 1$ extended solutions $(S^{(e_0,i,j,k)}, x^{(e_0,i,j,k)})$ such that $x_e^{(e_0,i,j,k)} = x_e^{(e_0,i,j)}$ for $e \in S^{(e_0,i,j)} \setminus \{e_0\}$ and $x_{e_0}^{(e_0,i,j,k)} = k$, $k = c_{e_0}, c_{e_0}+1, \ldots, d_{e_0}$. Let Q_E denote the set of the extended solutions $(S^{(e_0,i,j,k)}, x^{(e_0,i,j,k)})$, and let P_E denote the set of the corresponding points $(F_1(S^{(e_0,i,j,k)}, x^{(e_0,i,j,k)}), F_2(S^{(e_0,i,j,k)}, x^{(e_0,i,j,k)}))$.

Step 3. Taking the sets Q_E and P_E as an input, apply the algorithm of Kung et al. [27] to find the set of efficient solutions and the corresponding Pareto front for the problem BI-MIN-RISK.

Recall that $W = \max_{e\in E}\{d_e - c_e + 1\}$. Since the algorithm in Kung et al. [27] requires $O(|P_E|\log|P_E|)$ time and $|P_E| \le m|K|W$, the procedure **Bi-Enum-2** can be implemented to run in $O(R_1'' + m|K|W\log(m|K|W))$ time, where R_1'' is the time complexity of its Step 1. The fact that in Step 2 a solution is found for each point of the Pareto front (and, possibly, for some other points) justifies its correctness.

For the minimum spanning tree and shortest path cases of the problem BI-MIN-RISK, algorithms in Sections 3.2 and 3.3, respectively, can be used in Step 1 of the procedures **Bi-Enum-1** and **Bi-Enum-2**. The set case of this problem can be solved in $O(m^3 d_{\max})$ (resp. $O(m^2 W\log(m^2 W))$) time by applying the procedure **Bi-Enum-1** (resp. **Bi-Enum-2**). However, there exists a more efficient algorithm for this problem. Indeed, to find the set of efficient solutions of this problem, it is sufficient to solve the problem MIN-RISK-SET for each value $B \in \{1,2,\ldots,B^0\}$. Based on the results in Section 3.1, it can be done in $O(mB^0)$ time, that is, in $O(m^2 d_{\max})$ time.

6 | CONCLUSIONS, EXTENSIONS, AND SUGGESTIONS FOR FUTURE RESEARCH

We have proposed solution procedures for the problems MIN-RISK, REV-MIN-RISK, BI-MIN-RISK and their special cases. These problems are generalizations of the risk constrained and risk minimization problems introduced in Álvarez-Miranda

TABLE 1 Asymptotic running times of algorithms for problems with risk functions $f_e(x_e) = \frac{d_e - x_e}{d_e - c_e}$

Problem	This article	Earlier papers
MIN-RISK-SET	$O(m)$	–
REV-MIN-RISK-SET	$O(m)$	–
BI-MIN-RISK-SET	$O(m^2 d_{\max})$	–
MIN-RISK-SPAN	$O(mn(m\alpha(m,n) + n\log n))^{a}$	$O(m^2 \log m \log n(m + n\log n))$ [10]
REV-MIN-RISK-SPAN	$O(mn(m\alpha(m,n) + n\log n + \log W))^{a}$	$O(m^3 \log m \log(md_{\max}) \log n(m + n\log n))$ [1]
BI-MIN-RISK-SPAN	$O(mn(m\alpha(m,n) + n\log n + nd_{\max}))^{a}$ $O(mn(m\alpha(m,n) + n\log n + W\log(mnW)))^{a}$	–
MIN-RISK-PATH	$O(mn)$	$O(mn^3)$ [9]
REV-MIN-RISK-PATH	$O(mn\log W)$	$O(m^2n^3 \log(md_{\max}))$ [1]
BI-MIN-RISK-PATH	$O(mn^2 d_{\max})$ $O(mnW\log(mnW))$	–

[a] If all the values c_e, $e \in E$, are distinct and all the values d_e, $e \in E$, are distinct, then the term $mn(m\alpha(m,n) + n\log n)$ should be replaced by mn^2; see Comment 3.

et al. [1] and Chen et al. [9, 10]. Our results are inspired by the earlier studies, generalize them and extend solution procedures suggested earlier by introducing series of problems $P(e_0, i, j)$ and solving them by the most efficient existing algorithms. The developed linear time algorithms for the problems MIN-RISK-SET and REV-MIN-RISK-SET are asymptotically best possible for these problems.

Table 1 contains asymptotic running times of the algorithms suggested for the problems with the risk function defined in [1, 9, 10]. For the bicriteria problems, running time estimations of the procedures **Bi-Enum-1** and **Bi-Enum-2** are given. Note that the asymptotic running times of all our algorithms are lower than those of the earlier methods in [1, 9, 10].

Since all the risk functions f_e, $e \in E$, are arbitrary concave nonincreasing in our studies, all the results are applicable for the following maximization problem, denoted as MAX-SAFETY, provided that the following assumption is satisfied for functions g_e which replace functions f_e in the maximization problem.

Assumption 2. All the safety functions $g_e(x_e)$, $e \in E$, are nondecreasing, convex, and satisfy the equalities $g_e(c_e) = k_1$ and $g_e(d_e) = k_2$, where k_1 and k_2 are given real numbers such that $k_1 < k_2$.

Problem MAX-SAFETY.

$$\max_{(S,x)} \sum_{e \in S} g_e(x_e), \quad \text{subject to}$$
$$S \in \mathcal{S}(E),$$
$$\sum_{e \in S} x_e \le B,$$
$$x_e \in \{c_e, c_e + 1, \ldots, d_e\}, \quad e \in S.$$

In the future, it is interesting to adapt the solution procedures in this article for controllable risk problems defined on other combinatorial structures such as the bases of a matroid, Steiner trees of a graph, or subtrees of an out-tree; see Khachiyan et al. [25], Onn [31], Álvarez-Miranda et al. [2], Du and Hu [15], and Johnson and Niemi [23] for the definitions of these structures and the relevant results. Improving complexities of the developed algorithms is interesting as well. Another attractive task is to study risk functions f_e, $e \in E$, that do not respect Assumption 1.

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