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On five types of stability of the lexicographic variant of the combinatorial bottleneck problem

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Abstract — We consider the combinatorial vector minimax problem with ordered criteria. We formulate necessary and sufficient conditions for the five known types of stability of the problem which describe the behaviour of the lexicographic set with respect to perturbations of the initial data for the vector criterion.

1. INTRODUCTION

One of the known approaches to investigation of the problem on stability of vector problems of discrete optimisation is aimed at obtaining results of a qualitative character, namely, at finding the conditions under which the set of optimal (in certain sense) solutions of the problem possesses some given invariance property under influence of outside factors on initial data of the problem. The most part of the results in this direction is connected with obtaining necessary and sufficient conditions for the five known types of stability of the vector integer problems of linear and quadratic programming, which consist of finding the Pareto, Slater, or Smale optimal solutions (see, for example, [1–6]) and also with Boolean problems of sequential minimisation of moduli of linear functions [7].

Here we give analysis of the five types of stability (with respect to a vector criterion) of a multicriteria combinatorial problem with successive application of minimax criteria. As a result we obtain necessary and simultaneously sufficient conditions for each type of stability. Similar conditions for vector combinatorial bottleneck problem with Pareto optimality principle were obtained in [8].

2. DEFINITIONS, NOTATION, AND PROPERTIES

Consider the vector variant of the well-known combinatorial bottleneck problem. On the set $N_m = \{1, 2, \dots, m\}$, $m \geq 2$, let some system of nonempty subsets $T \subseteq 2^{N_m} \setminus \{\varnothing\}$, $|T| \geq 2$, be given. The elements of the set T are usually referred to as trajectories. Let the components of the vector-function

$$f(t, A) = (f_1(t, A), f_2(t, A), \dots, f_n(t, A)), \qquad n \geq 1,$$

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defined on T, be the minimax criteria

$$f_i(t, A) = \max_{j \in t} a_{ij} \underset{t \in T}{\longrightarrow} \min, \qquad i \in N_n,$$

where $A = [a_{ij}]$ is an $n \times m$ matrix with elements belonging to $\mathbf{R}$.

The vector (n-criteria) combinatorial problem $Z^n(A)$, $n \geq 1$, is the problem to find the lexicographic set (the set of lexicographic optima)

$$L^n(A) = \{t \in T \colon \forall t' \in T \ (t \underset{A}{\not\succ} t')\},$$

where $\underset{A}{\not\succ}$ is the negation of the binary lexicographic relation $\underset{A}{\succ}$ defined on the set of trajectories T by the formula

$$t \underset{A}{\succ} t' \iff \exists k \in N_n \ (f_k(t, A) > f_k(t', A) \ \& \ k = \min\{i \in N_n \colon f_i(t, A) \neq f_i(t', A)\}).$$

We observe that many classical extremal problems on graphs (travelling salesman problem, problems on matchings, on spanning sets, etc.), various scheduling problems and problems of Boolean programming are covered by the scheme of scalar (one-criterion) combinatorial problems (with linear, maximin, minimax, and other criteria, see [1, 9–11]).

We will use the notation $\overline{L^n}(A) = T \setminus L^n(A)$.

Property 1. *If* $t \underset{A}{\succ} t'$, *then* $t \in \overline{L^n}(A)$.

Property 2. *If* $t \underset{A}{\succ} t'$, *then* $t' \underset{A}{\not\succ} t$.

It is known (see, for example, [12]) that the problem $Z^n(A)$ can be treated as the problem of optimisation on successively applied criteria, that is, the lexicographic set $L^n(A)$ can be defined as a result of solution of the sequence of n scalar problems

$$L_i^n(A) = \operatorname{Arg\,min}\{f_i(t, A) \colon t \in L_{i-1}^n(A)\}, \qquad i \in N_n,$$

where $L_0^n(A) = T$, $\operatorname{Arg\,min}\{\cdot\}$ is the set of all optimal trajectories of the corresponding minimisation problem. Hence it follows that the relations

$$T \supseteq L_1^n(A) \supseteq L_2^n(A) \supseteq \ldots \supseteq L_n^n(A) = L^n(A)$$

are true.

It is obvious that for any matrix $A \in \mathbf{R}^{n \times m}$ the set $L^n(A)$ is not empty.

It is clear that $t \underset{A}{\succ} t'$ if $t \notin L_k^n(A)$ and $t' \in L_k^n(A)$ for some $k \in N_n$. Therefore, the following property is true.

Property 3. *Let* $k \in N_n$ *be chosen in such a way that* $f_k(t, A) > f_k(t', A)$ *and* $t' \in L_k^n(A)$. *Then* $t \underset{A}{\succ} t'$.

It is clear that for any trajectory t and any $n \times m$ matrix B with nonnegative elements ($B \in \mathbf{R}_+^{n \times m}$) the inequalities

$$f_i(t, A) \leq f_i(t, A + B), \qquad i \in N_n,$$

are true. Thus, the following property is valid.

Property 4. *Let $k \in N_n$. If a trajectory $t \in L_k^n(A)$ and a matrix $B^0 \in \mathbf{R}_+^{n\times m}$ are chosen in such a way that*

$$f_i(t, A) = f_i(t, A + B^0), \qquad i \in N_k,$$

then $t \in L_k^n(A + B^0)$.

Let us give the definitions of the five most widespread types of stability (see, for example, [1–8, 13–16].

A problem $Z^n(A)$, $n \geq 1$, is called stable (with respect to perturbance of elements of the matrix A) if

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (L^n(A + B) \subseteq L^n(A)).$$

Here

$$\Xi(\varepsilon) = \{B \in \mathbf{R}^{n\times m} : \|B\| < \varepsilon\}$$

is the set of perturbing matrices,

$$\|B\| = \max\{|b_{ij}| : (i, j) \in N_n \times N_m\},$$

$B = [b_{ij}]$. The problem $Z^n(A + B)$ will be referred to as the perturbed problem.

Thus, the problem is stable if any sufficiently small perturbations of the initial data of the problem do not lead to emergence of new lexicographic optima. So the stability can be treated as a discrete analogue of the Hausdorff upper semicontinuity at the point A of many-valued optimal mapping

$$L^n : \mathbf{R}^{n\times m} \to 2^T, \tag{1}$$

which puts the lexicographic set of problems into correspondence with each matrix in $\mathbf{R}^{n\times m}$.

Note that a numerical estimate of the admissible perturbations of the initial data (the so-called stability radius) of the scalar bottleneck problem was considered in a series of papers (see, for example, [10, 18, 19]).

Weakening the requirement that new classes of lexicographic optima do not emerge, we arrive at the notion of strong stability of a problem. A problem $Z^n(A)$ is called strongly stable if

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (L^n(A) \cap L^n(A + B) \neq \varnothing).$$

Thus, this type of stability describes the situation where the lexicographic set of the problem has an empty intersection with a similar set of any perturbed problem. It is easy to see that the problem $Z^n(A)$ is strongly stable if it is stable. It will be shown below (see Theorem 1) that the reverse assertion is also true.

A problem $Z^n(A)$ is called quasistable if

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (L^n(A) \subseteq L^n(A + B)).$$

In other words, quasistability characterises the case where all trajectories of a lexicographic set do not lose the property of being optimal under small enough perturbations of the initial data of the problem. Thus, quasistability of the problem $Z^n(A)$ is a discrete analogue of the Hausdorff lower semicontinuity at the point A of the many-valued optimal mapping (1).

Weakening the requirement that the whole lexicographic set of the problem is preserved, we arrive at the notion of strong quasistability. This type of stability is related with the presence of at least one trajectory belonging to the lexicographic sets of all perturbed problems under quite small perturbations of the initial data. Thus, a problem $Z^n(A)$ is called strongly quasistable if

$$\exists\varepsilon > 0 \quad \exists t^0 \in L^n(A) \quad \forall B \in \Xi(\varepsilon) \quad (t^0 \in L^n(A + B)).$$

It is obvious that any quasistable problem is strongly quasistable and any strongly quasistable problem is strongly stable.

Finally, a problem $Z^n(A)$ is called steady if

$$\exists\varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (L^n(A) = L^n(A + B)).$$

Thus, the property of a problem $Z^n(A)$ to be steady is the discrete analogue of the Hausdorff continuity at the point A of the many-valued optimal mapping (1). It is obvious that a problem $Z^n(A)$ is stable if and only if it is stable and quasistable simultaneously.

Note that the defined above types of stability were called T_1–T_5-stabilities in [2–6].

For any nonempty subset $I \subseteq N_n$, on the set of trajectories T of the set $Z^n(A)$ we introduce the binary relations

$$t \underset{I,A}{\geq} t' \iff \forall i \in I \quad (f_i(t, A) \geq f_i(t', A)),$$

$$t \underset{I,A}{>} t' \iff \forall i \in I \quad (f_i(t, A) > f_i(t', A)),$$

$$t \underset{I,A}{\vdash} t' \iff \forall i \in I \quad (N_i(t, A) \supseteq N_i(t', A)),$$

where

$$N_i(t, A) = \{j \in t : a_{ij} = f_i(t, A)\}.$$

It is obvious that the following properties are valid.

Property 5. *If* $t \underset{I,A}{\vdash} t'$, *then there exists* $\varepsilon > 0$ *such that for any perturbing matrix* $B \in \Xi(\varepsilon)$ $t \underset{I,A+B}{\geq} t'$.

Property 6. *If* $t \underset{N_n,A}{\geq} t'$, *then* $t' \underset{A}{\succeq} t$.

Taking into account Properties 5 and 6 and the continuity of the function $f(t, A)$ on the set of parameters $\mathbf{R}^{n\times m}$, we conclude that the following two properties are valid.

Property 7. *If* $t \underset{N_n,A}{\vdash} t'$, *then*

$$\exists\varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t' \underset{A+B}{\succeq} t).$$

Property 8. *If trajectories t and t' satisfy any of the conditions*

(i) $t \underset{1,A}{>} t'$,

(ii) $\exists k \in N_{n-1}\ (t \underset{N_k,A}{\vdash} t' \,\&\, t \underset{k+1,A}{>} t')$,

then

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t \underset{A+B}{\succ} t'). \tag{2}$$

Further we set

$$\begin{aligned} M(t) &= \{i \in N_n : t \in L_i^n(A)\}, \\ \sigma(t) &= |M(t)|, \\ V^n(A) &= \{t \in L_1^n(A) : \exists t' \in L^n(A)\ (t \underset{M(t),A}{\vdash} t')\}, \\ U^n(A) &= \{t \in L^n(A) : \forall i \in N_n\ \forall t' \in L_i^n(A)\ (t' \underset{i,A}{\vdash} t)\}. \end{aligned}$$

It is clear that the set $U^n(A)$ can be an arbitrary (maybe empty) subset of the lexicographic set $L^n(A)$.

In view of reflexivity of the binary relation $\underset{i,A}{\vdash}$, the inclusion $L^n(A) \subseteq V^n(A)$ is true, and therefore, $L^n(A) = V^n(A)$ if $L^n(A) = L_1^n(A)$. It is also easy to see that for $t \in L_1^n(A)$ the set $M(t)$ coincides with the set $N_{\sigma(t)}$, where $1 \le \sigma(t) \le n$. We observe that $\sigma(t) = n$ only if $t \in L^n(A)$. Therefore, the following property is valid.

Property 9. *Let $t \in L_1^n(A) \setminus L^n(A)$ and $t' \in L^n(A)$. Then $1 \le \sigma(t) < n$ and $t \underset{\sigma(t)+1,A}{>} t'$.*

It is easy to see that the following properties are also valid.

Property 10. *If $t \in L_1^n(A) \setminus V^n(A)$, then $N_m \setminus t \neq \varnothing$.*

Property 11. *If $t \in L^n(A)$ and $t' \in U^n(A)$, then $t \underset{N_n,A}{\vdash} t'$*

3. AUXILIARY RESULTS

Lemma 1. *If $t \in L_1^n(A) \setminus V^n(A)$, then*

$$\forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad \forall t' \in L^n(A) \quad (t' \underset{A+B^0}{\succ} t). \tag{3}$$

Proof. Let $t \in L_1^n(A) \setminus V^n(A)$. Then for any trajectory $t' \in L^n(A)$ there is $k = k(t') \in M(t) = N_{\sigma(t)}$ such that $N_k(t, A) \not\supseteq N_k(t', A)$ and $t \in L_k^n(A)$. Then $t' \setminus t \neq \varnothing$. Let $p \in N_k(t', A) \setminus N_k(t, A)$. Then

$$f_k(t, A) = f_k(t', A) = a_{kp}. \tag{4}$$

Further, setting $\varepsilon > 0$ and taking into account Property 10, we construct the perturbing matrix $B^0 = [b^0_{ij}] \in \mathbf{R}^{n\times m}$ by the rule

$$b^0_{ij} = \begin{cases} \delta & \text{if } i \in M(t),\ j \in N_m \setminus t, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon$. Then it is clear that

$$f_i(t, A) = f_i(t, A + B^0), \qquad i \in M(t).$$

Therefore, it follows from Property 4 and the inclusion $t \in L^n_k(A)$ that $t \in L^n_k(A + B^0)$. Moreover, using equality (4), we conclude that

$$\begin{aligned} f_k(t', A + B^0) &= \max\{a_{kj} + b^0_{kj} : j \in t'\} = \max\{a_{kj} + b^0_{kj} : j \in t' \setminus t\} \\ &= a_{kp} + \delta > a_{kp} = f_k(t', A) = f_k(t, A) = f_k(t, A + B^0). \end{aligned}$$

Thus, formula (3) follows from Property 3. Lemma 1 is proved.

Lemma 2. *If* $L^n_1(A) = V^n(A)$, $t \in \overline{L^n}(A)$, *then there exists a trajectory* $t^0 \in L^n(A)$ *such that*

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t \underset{A+B}{\succ} t^0). \tag{5}$$

Proof. Consider two possible cases for a trajectory $t \in \overline{L^n}(A)$.

Case 1: $t \in L^n_1(A)$. Then $t \in V^n(A)$, and therefore, there exists a trajectory $t^0 \in L^n(A)$ such that the relation $t \underset{M(t),A}{\vdash} t^0$ is fulfilled. In addition, taking regard for the inclusion $t \in \overline{L^n}(A) \cap L^n_1(A) = L^n_1(A) \setminus L^n(A)$ and Property 9, we conclude that $t \underset{\sigma(t)+1,A}{>} t^0$. This relation and Property 8 imply that

$$\exists \varepsilon_1 > 0 \quad \forall B \in \Xi(\varepsilon_1) \quad (t \underset{A+B}{\succ} t^0).$$

Case 2: $t \in T \setminus L^n_1(A)$. Then for any trajectory $t^0 \in L^n(A)$ the relation $t \underset{1,A}{>} t^0$ is fulfilled. Therefore it follows from Property 8 that

$$\exists\, \varepsilon_2 > 0 \ \forall B \in \Xi(\varepsilon_2) \ (t \underset{A+B}{\succ} t^0).$$

Upon analysing both of the cases, we see that formula (5) is true. Lemma 2 is proved.

Lemma 3. *If* $t \in U^n(A)$ *and* $t' \in T$, *then*

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t \underset{A+B}{\succeq} t'). \tag{6}$$

Proof. Let $t \in U^n(A)$. Consider two possible cases for the trajectory t'.

Case 1: $t' \in L^n_1(A)$. Let $t' \in L^n(A)$. Then, by Property 11, the relation $t' \underset{N_n,A}{\vdash} t$ is true, so, with the use of Property 7, we obtain (6).

Let $t' \in L_1^n(A) \setminus L^n(A)$. Then there exists $k = k(t') \in N_n \setminus \{1\}$ such that $t' \notin L_k^n(A)$ and $t' \in L_i^n(A)$ for $i \in N_{k-1}$. Therefore $t' \underset{N_{k-1},A}{\vdash} t$ and $t' \underset{k,A}{>} t$. It follows from these facts and Property 8 that the formula

$$\exists \varepsilon > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t' \underset{A+B}{\succ} t),$$

is true. This formula and Property 2 imply (6).

Case 2: $t' \in T \setminus L_1^n(A)$. Then the relation $t' \underset{1,A}{>} t$ is true. This fact and Properties 2 and 8 imply (6).

Lemma 3 is thus proved.

Lemma 4. *If* $t \in L^n(A) \setminus U^n(A)$, *then*

$$\exists t^0 \in T \quad \forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad (t \underset{A+B^0}{\succ} t^0). \tag{7}$$

Proof. Since $t \notin U^n(A)$, there exists some $k \in N_n$ and $t^0 \in L_k^n(A)$ such that $N_k(t^0, A) \not\supseteq N_k(t, A)$ and $t \in L_k^n(A)$. Let $p \in N_k(t, A) \setminus N_k(t^0, A)$. Then $f_k(t, A) = f_k(t^0, A) = a_{kp}$. Therefore, choosing $\varepsilon > 0$ and constructing elements of the perturbing matrix $B^0 = [b_{ij}^0] \in \mathbf{R}^{n\times m}$ by the rule

$$b_{ij}^0 = \begin{cases} \delta, & \text{if } i = k,\ j = p, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon$, we see that the relations

$$\begin{aligned} f_i(t^0, A) &= f_i(t^0, A + B^0), \qquad i \in N_k, \\ f_k(t, A + B^0) &= \max\{a_{kj} + b_{kj}^0 \colon j \in t\} = a_{kp} + \delta > a_{kp} = f_k(t, A) = f_k(t^0, A) \\ &= f_k(t^0, A + B^0). \end{aligned}$$

are true. Hence it follows that $t^0 \in L_k^n(A + B^0)$ by Property 4 and $t \underset{A+B^0}{\succ} t^0$ by Property 3. Thus, formula (7) is true. Lemma 4 is proved.

4. ANALYSIS OF THE FIVE TYPES OF STABILITY

Theorem 1. *For the vector problem* $Z^n(A)$, $n \geq 1$, *the following assertions are equivalent:*

(1) *the problem* $Z^n(A)$ *is stable,*

(2) *the problem* $Z^n(A)$ *is strongly stable,*

(3) $L_1^n(A) = V^n(A)$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Let us prove that (2) $\Rightarrow$ (3). Assume the contrary, that is, assume that there exists a problem $Z^n(A)$ which is strongly stable, but $L_1^n(A) \neq V^n(A)$. Let $t \in L_1^n(A) \setminus V^n(A)$. Then, by virtue of Lemma 1, in view of Property 1

$$\forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad \forall t' \in L^n(A) \quad (t' \in \overline{L^n}(A + B^0)).$$

Hence we obtain the relation

$$\forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad (L^n(A) \cap L^n(A + B^0) = \varnothing),$$

which contradicts the strong stability of the problem $Z^n(A)$.

Let us prove that (3) $\Rightarrow$ (1). It is clear that if $L^n(A) = T$, then the problem $Z^n(A)$ is stable. Let $\overline{L^n}(A) \neq \varnothing$. Then by virtue of Lemma 2 and Property 1 for any trajectory $t \in \overline{L^n}(A)$

$$\exists \varepsilon = \varepsilon(t) > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t \in \overline{L^n}(A + B)).$$

Therefore, it is easy to see that

$$\exists \varepsilon^* > 0 \quad \forall B \in \Xi(\varepsilon^*) \quad (\overline{L^n}(A) \subseteq \overline{L^n}(A + B)).$$

for

$$\varepsilon^* = \min\{\varepsilon(t) : t \in \overline{L^n}(A)\}.$$

Thus, the problem $Z^n(A)$ is stable. Theorem 1 is proved.

Note that the equivalence of the notions of stability and strong stability of the problem $Z^n(A)$ holds not for all vector problems of discrete optimisation. For example, in [17] it is shown that any vector problem of integer linear programming with Pareto optimality principle is strongly stable, but it is well known (see, for example, [1, 2]) that such a problem is stable if and only the Pareto set and Slater set coincide.

Theorem 1 has the following equivalent formulation.

Theorem 2. *The problem $Z^n(A)$ is stable (strongly stable) is and only if*

$$L^n(A) \neq L_1^n(A) \implies \forall t \in L_1^n(A) \setminus L^n(A) \quad \exists t^0 \in L^n(A) \quad (t \underset{M(t),A}{\vdash} t^0).$$

The following assertion is a direct corollary of Theorem 2.

Corollary 1. *If any of the following conditions is fulfilled:*

(1) $L^n(A) = L_1^n(A)$,

(2) $|\{y \in \mathbf{R}^n : y = f(t, A),\ t \in L_1^n(A)\}| = 1$,

(3) $\forall t \in \overline{L^n}(A)\ \forall t' \in L^n(A)\ (t \underset{1,A}{>} t')$,

(4) $\forall t \in \overline{L^n}(A)\ \exists t' \in T\ (t \underset{1,A}{>} t')$

then the problem $Z^n(A)$ is stable and strongly stable.

The following example shows that the equality $L^n(A) = L_1^n(A)$ is not a necessary condition for stability (strong stability).

Example 1. Let

$$n = 2, \quad m = 2, \quad T = \{t_1, t_2\}, \quad t_1 = \{1\}, \quad t_2 = \{1, 2\}, \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$f(t_1, A) = (1, 1), \quad f(t_2, A) = (1, 2), \quad L^2(A) = \{t_1\}, \quad L_1^2(A) = T.$$

It is easily seen that

$$M(t_1) = \{1, 2\}, \quad M(t_2) = \{1\}, \quad t_1 \underset{M(t_1),A}{\vdash} t_1, \quad t_2 \underset{M(t_2),A}{\vdash} t_1.$$

Therefore,

$$V^2(A) = L_1^2(A).$$

Thus, according to Theorem 1, the problem $Z^2(A)$ is stable, but $L^2(A) \neq L_1^2(A)$.

Note that it is shown in [13–15] that the conditions given in Corollary 1 are not only sufficient, but they are necessary for stability and strong stability of a vector combinatorial problem if the criteria are linear.

In the scalar case ($n = 1$), Theorem 1 takes the following form.

Corollary 2. *The scalar problem $Z^1(A)$ is stable and strongly stable for any vector $A \in \mathbf{R}^m$.*

Theorem 3. *The vector problem $Z^n(A)$, $n \geq 1$, is quasistable if and only if*

$$L^n(A) = U^n(A).$$

Proof. Let us prove the necessity. Let the problem $Z^n(A)$ be quasistable but $L^n(A) \neq U^n(A)$. Then there exists a trajectory $t \in L^n(A) \setminus U^n(A)$ for which, by Lemma 4 and Property 1,

$$\forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad (t \in \overline{L^n}(A + B^0)).$$

Hence it follows that

$$\forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad (L^n(A) \not\subseteq L^n(A + B^0)),$$

which contradicts the quasistability of the problem $Z^n(A)$.

Let us prove the sufficiency. Let $L^n(A) = U^n(A)$ and $t \in U^n(A)$. Then by Lemma 3 for any trajectory $t' \in T$

$$\exists\, \varepsilon(t') > 0 \;\; \forall B \in \Xi(\varepsilon(t')) \;\; (t \underset{A+B}{\succeq} t').$$

Thus, for any trajectory $t \in L^n(A)$ there exists $\varepsilon = \varepsilon(t) > 0$ such that $t \in L^n(A + B)$ for any perturbing matrix $B \in \Xi(\varepsilon)$. Therefore, setting

$$\varepsilon^* = \min\{\varepsilon(t) : t \in L^n(A)\},$$

we obtain

$$\exists \varepsilon^* > 0 \quad \forall B \in \Xi(\varepsilon^*) \quad (L^n(A) \subseteq L^n(A + B)).$$

Thus, the problem $Z^n(A)$ is quasistable. Theorem 2 is proved.

It is obvious that Theorem 3 can be represented in the following form.

Theorem 4. *The vector problem $Z^n(A)$ is quasistable if and only if*

$$\forall t \in L^n(A) \quad \forall i \in N_n \quad \forall t' \in L_i^n(A) \quad (t' \underset{i,A}{\vdash} t).$$

Corollary 3. *The scalar problem $Z^1(A)$, $A \in \mathbf{R}^m$, is quasistable if and only if*

$$\forall t, t' \in L^1(A) \qquad (N_1(t, A) = N_1(t', A)).$$

Here $L^1(A)$ is the set of optimal trajectories of the scalar problem $Z^1(A)$.

Theorem 5. *The vector problem $Z^n(A)$, $n \geq 1$, is strongly quasistable if and only if $U^n(A) \neq \varnothing$.*

Proof. Let us prove the necessity. Let $U^n(A) = \varnothing$. Then, by Lemma 4 and Property 1, for any trajectory $t \in L^n(A)$

$$\forall \varepsilon > 0 \quad \exists B^0 \in \Xi(\varepsilon) \quad (t \in \overline{L^n}(A + B^0)).$$

Therefore, the problem $Z^n(A)$ is not strongly quasistable.

Let us prove the sufficiency. Let $t \in U^n(A)$. Then by Lemma 3 for any trajectory $t' \in T$

$$\exists \varepsilon = \varepsilon(t') > 0 \quad \forall B \in \Xi(\varepsilon) \quad (t \underset{A+B}{\succeq} t').$$

According to the definition of the lexicographic set, hence it follows that

$$\exists \varepsilon^* > 0 \quad \forall B \in \Xi(\varepsilon^*) \quad (t \in L^n(A + B)),$$

where

$$\varepsilon^* = \min\{\varepsilon(t') : t' \in T\}.$$

Thus, the problem $Z^n(A)$ is strongly quasistable. Theorem 5 is proved.

Corollary 4. *The scalar problem $Z^1(A)$ is strongly quasistable if and only if*

$$\exists t^0 \in L^1(A) \quad \forall t \in L^1(A) \quad (t \underset{1,A}{\vdash} t^0).$$

Theorems 1 and 3 imply the following assertion.

Theorem 6. *The problem $Z^n(A)$, $n \geq 1$, is steady if and only if*

$$V^n(A) = L_1^n(A), \qquad U^n(A) = L^n(A).$$

Theorems 1–6 imply the following assertion.

Corollary 5. *If $|L_1^n(A)| = 1$, then the problem $Z^n(A)$ is simultaneously stable, strongly stable, quasistable, strongly quasistable, and steady.*

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