Discrete Applied Mathematics 349 (2024) 106–112

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

A single representative min–max–min robust selection problem with alternatives and budgeted uncertainty

Nadia Brauner [a], Evgeny Gurevsky [b,*], Mikhail Y. Kovalyov [c]

[a] G-SCOP, Université Grenoble Alpes, France
[b] LS2N, Université de Nantes, France
[c] United Institute of Informatics Problems, NASB, Minsk, Belarus

ARTICLE INFO

Article history:
Received 25 August 2023
Received in revised form 6 February 2024
Accepted 6 February 2024
Available online xxxx

Keywords:
Discrete optimization
Uncertainty
Selection problem
Min–max–min
Two-stage
Robustness

ABSTRACT

A robust two-stage problem of selecting a single minimum cost representative out of n candidates is studied. Each candidate is associated with an uncertain cost that is described by a lower bound and a deviation from it. In the first stage, at most k representatives have to be selected. After that, an adversary distributes the worst costs to all representatives so that the sum of the cost deviation ratios (the uncertainty budget) does not exceed a given upper bound. In the second stage, the cost of the cheapest representative is paid. An $O(n^2 \log n)$ time algorithm is proposed for this problem.

1. Introduction

We study a robust discrete optimization problem with *budgeted uncertainty*. A short literature review on this topic is given in the next section. The studied problem is a special case of the problem introduced by Chassein and Goerigk [15]. In their terminology, the problem can be formulated as follows. Consider elements of a set $N = \{1, \ldots, n\}$, which we call *representatives*. Each representative i is associated with an uncertain cost c_i that belongs to a given interval $[\underline{c}_i, \underline{c}_i + d_i]$ of non-negative rational numbers. Introduce vector notation $v = (v_1, \ldots, v_n)$ for any v. Denote by $X^{(1)}$ the set of all n-dimensional 0–1 vectors x with a single 1, $|X^{(1)}| = n$. Let Γ be a given non-negative rational number. Consider a given number $k \in N$ and the cost vector set U with budgeted uncertainty:

$$U = \left\{ c \in \mathbb{R}^n_{\geq} \mid c_i \in [\underline{c}_i, \underline{c}_i + d_i],\ \ i \in N,\ \sum_{i \in N} \frac{c_i - \underline{c}_i}{d_i} \leq \Gamma \right\}.$$

The problem studied in this paper is denoted as 1-Min-Max-Min. It can be formulated as follows.

Problem 1-Min-Max-Min:

$$\min_{X \subseteq X^{(1)}, 1 \leq |X| \leq k} \max_{c \in U} \min_{x \in X} \sum_{i \in N} c_i x_i. \tag{1}$$

* Corresponding author.
E-mail address: evgeny.gurevsky@univ-nantes.fr (E. Gurevsky).

https://doi.org/10.1016/j.dam.2024.02.005
0166-218X/

It might also be convenient to consider an equivalent formulation of this problem. Define set of the cost deviation ratios as

$$Z = \left\{ z \mid 0 \le z_i \le 1, \sum_{i \in N} z_i \le \Gamma \right\}.$$

An equivalent formulation, in which variables c_i are replaced with z_i, is

$$\min_{X \subseteq X^{(1)}, 1 \le |X| \le k} \max_{z \in Z} \min_{x \in X} \sum_{i \in N} (\underline{c}_i + d_i z_i) x_i.$$

In the problem 1-Min-Max-Min, a set X of at most k alternative solutions (single representative selections, 0–1 vectors with a single one) has to be determined in the first stage, corresponding to the outer min in (1), with the aim of minimizing the cost of the final solution (single representative). In the second stage, the adversary, corresponding to the max in (1), distributes the budgeted uncertain cost over all the representatives so that the cheapest solution among the k chosen ones has the largest cost.

The 1-Min-Max-Min problem can be used for modeling the min-cost choice of a single Logistics Provider (LP) out of n candidates for a future shipment. Preliminary agreements can be signed with at most k LPs, the penalty for breaking these agreements is negligible comparing to the future shipment cost, and the shipment cost is subject to the budgeted uncertainty such that this cost belongs to a given LP-dependent interval and the total relative cost fluctuation in these intervals is historically limited by Γ.

Following [15], we assume that $\Gamma \ge 1$. If $\Gamma < 1$, then the cost deviations can be re-set as $d_i := \Gamma d_i$, $i \in N$, and Γ can be re-set to 1. Assume also without loss of generality that $\Gamma < k$. If $\Gamma \ge k$, then setting $c_i = \underline{c}_i + d_i$ for i such that $x_i = 1$ and $x \in X$ is optimal for the adversary for any $X \subseteq X^{(1)}$, $1 \le |X| \le k$. Therefore, the problem becomes trivial in this case: optimal set X consists of the single vector x with $x_i = 1$, where index i corresponds to the minimal cost upper bound $\underline{c}_i + d_i$.

Chassein and Goerigk [15] study a generalization of the problem 1-Min-Max-Min in which $p \ge 1$ representatives have to be selected at once, that is, each feasible 0–1 vector contains exactly p number of ones. We denote such generalization of the problem 1-Min-Max-Min as p-Min-Max-Min. Chassein and Goerigk call the problem p-Min-Max-Min as the selection problem and propose a solution approach that is to solve $O(n^{2k-1})$ linear programming problems. They also describe an $O(n^4 \log n)$ time algorithm for a special case of $k = 2$ (Section 4.3 in [15]). The existence of a polynomial-time algorithm for any of the problems p-Min-Max-Min and 1-Min-Max-Min, in which k is an arbitrary number, remained unknown. In Section 3 we propose an $O(n^2 \log n)$ time algorithm for the problem 1-Min-Max-Min, in which k is an arbitrary number.

2. Literature review

Optimization problems with budgeted uncertainty have become a popular research topic recently. Below we give a short review of several relevant publications. The concept of budgeted uncertainty was introduced by Bertsimas and Sim [7,8]. According to this concept, for a cost minimization problem, the cost vector can be any vector from a set of scenarios. This set includes cost vectors which are within a given range, called uncertainty budget, from a nominal cost vector. The bibliography of this line of research includes the following works.

Poss [27] introduced an extension of the budgeted uncertainty concept in [7,8], the so-called variable budgeted uncertainty, and applied it to an uncertain knapsack problem. Goerigk and Hartisch [19] proposed a generalization of the min–max regret concept which they called balanced regret, and applied it to some general combinatorial problems with more attention paid to a selection problem.

Alves Pessoa et al. [4] and Di Puglia Pugliese et al. [18] studied a resource constrained shortest path problem with budgeted uncertainty. Agra et al. [1] applied the budgeted uncertainty concept to a lot-sizing problem, a traveling salesman problem with time windows, a scheduling problem and an inventory routing problem, and Ayoub and Poss [5] — to the telecommunication network design under demand uncertainty.

Single machine scheduling problems with budgeted uncertainty were studied by Ales et al. [3], Silva et al. [28] and Bougeret et al. [13], parallel machine scheduling problems — by Bougeret et al. [11,12] and Albers and Janke [2], flow shop scheduling problems — by Levorato et al. [24,25], and a cyclic scheduling problem — by Hamaz et al. [23]. Bruni et al. [14], Bold and Goerigk [9,10] and Bendotti et al. [6] concentrated on a resource-constrained project scheduling problem with budgeted uncertainty. Mattia and Poss [26] applied budgeted uncertainty to solving a robust network loading problem.

Chassein et al. [16] introduced several robust item selection problems with budgeted uncertainty. They developed combinatorial polynomial time algorithms for special cases and suggested mixed-integer programming (MIP) formulations for the most general continuous and discrete cases. The MIP formulations for the continuous problems were proved to be polynomially solvable. Goerigk et al. [22] proved that the discrete problems are NP-hard in the ordinary sense. Chassein and Goerigk [15], Chassein et al. [17] and Goerigk et al. [20,21] investigated the same budgeted uncertainty setting as in the problem 1-Min-Max-Min on the examples of a knapsack problem, an unconstrained combinatorial optimization problem, a matroid maximization problem, a selection problem and a shortest path problem. Their $O(n^4 \log n)$ time algorithm for the selection problem can be used to solve the special case $k = 2$ of the 1-Min-Max-Min problem.

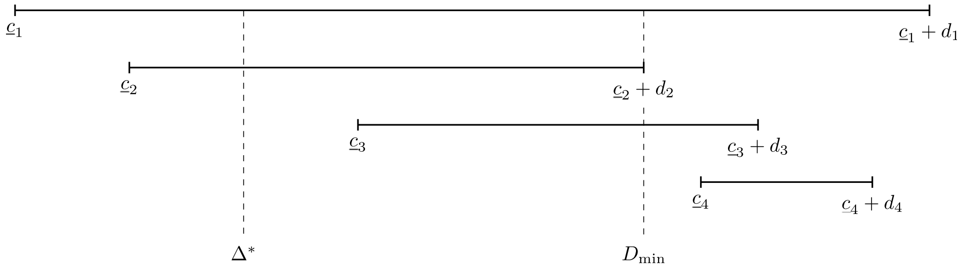

Fig. 1. Example of 1-Min-Max-Min, in which $n = 4$, $i^0 = 2$ and $i^1 = 3$.

3. Properties and $O(n^2 \log n)$ time algorithm for 1-min-max-min

We begin with several useful observations on the properties of the problems close to the problem 1-Min-Max-Min. Denote problem 1-Min-Max-Min with a fixed set $X \subseteq X^{(1)}$, $1 \le |X| \le k$, as 1-Max-Min(X).

Observation 1. *If problem* 1-Max-Min(X) *is solvable in* $O(T)$ *time, then the respective problem* 1-Min-Max-Min *is solvable in* $O(n^k T)$ *time.*

Proof. Follows from the fact that the number of subsets $X \subseteq X^{(1)}$ such that $|X| \le k$ is $\sum_{i=1}^{k} \binom{n}{i} = O(n^k)$. □

It is convenient to denote a solution of the problem p-Min-Max-Min as an ordered triple (X, c, x), where $X \subseteq X^{(p)}$, $c \in U$ and $x \in X$. In the robust optimization literature, only the first-stage decision X is called a solution, and c and x are called policies of the adversary and the decision maker, respectively. The adversarial policy c depends on X and the second-stage decision x depends on c and X.

Observation 2. *If the requirement of selecting exactly p representatives is replaced by the requirement of selecting at least p representatives, then there exists an optimal solution of the problem* p-Min-Max-Min, *in which exactly p representatives are selected.*

Proof. Consider an optimal solution (X^0, c^0, x^0) of the problem p-Min-Max-Min in which at least p representatives have to be selected. Update set X^0 so that if $\sum_{i=1}^{n} x_i = h > p$ for a vector $x \in X^0$, then re-set arbitrary $h - p$ elements of this vector from $x_i = 1$ to $x_i = 0$. Denote the updated set as X'. We have now $\sum_{i=1}^{n} x_i = p$ for each $x \in X'$. Consider a vector $x \in X^0$ and the respective updated vector $x' \in X'$. Observe that $\sum_{i \in N} c_i x'_i \le \sum_{i \in N} c_i x_i$ holds for any $c \in U$, which implies that a solution (X', c', x'') is also optimal with respect to the problem p-Min-Max-Min, in which at least p representatives have to be selected, for a certain $c' \in U$ and $x'' \in X'$. □

Corollary 1. *The "unconstrained problem" (Section 4.1 in Chassein and Goerigk [15]), in which no constraint is placed on the set of 0–1 vectors X selected in the first stage of the problem* p-Min-Max-Min, *admits a trivial optimal solution with* $X = \{(0, \dots, 0)\}$. *The respective adversarial optimal policy is to choose any cost vector* $c \in U$.

Denote an optimal solution of the problem 1-Min-Max-Min as an ordered triple (X^*, c^*, x^*), where $X^* \subseteq X^{(1)}$, $c^* \in U$ and $x^* \in X^*$, and denote its optimal objective function value, which is calculated by (1), as Δ^*. Denote $D_{\min} := \min_{i \in N}\{\underline{c}_i + d_i\}$. For a given set $X \subseteq X^{(1)}$, introduce the set of indices (alternative representatives)

$$N(X) = \{i \in N \mid x_i = 1 \text{ for some } x \in X\}.$$

There is a one-to-one correspondence between elements of the sets X and $N(X)$. Re-number the representatives so that $\underline{c}_1 \le \dots \le \underline{c}_n$, and after the re-numbering, denote by i^0 the smallest index such that $\underline{c}_{i^0} + d_{i^0} = D_{\min}$ and by i^1 the largest index such that $\underline{c}_{i^1} < D_{\min}$ if it exists. If $\underline{c}_i \ge D_{\min}$ for all $i \in N$, then define $i^1 = i^0$. We proceed with proving important properties of the problems 1-Min-Max-Min and 1-Max-Min(X). The proofs are supported by Fig. 1.

Lemma 1. *There exists an optimal solution of* 1-Min-Max-Min *that satisfies the following properties.*
(i) If $\Delta^* = D_{\min} = \underline{c}_{i^0} + d_{i^0}$, *then* (X^*, c^*, x^*) *is an optimal solution, where* $X^* = \{x^*\}$, x^* *is the 0–1 vector with single 1 in position* i^0, $c^*_{i^0} = \underline{c}_{i^0} + d_{i^0}$ *and* $c^*_i = \underline{c}_i$, $i \in N \setminus \{i^0\}$.
(ii) $\Delta^* \le D_{\min}$.
(iii) If $\Delta^* < \underline{c}_i$, *then* $x_i = 0$ *for all* $x \in X^*$, *that is,* $i \in N \setminus N(X^*)$.
(iv) If $\Delta^* < D_{\min}$, *then* $N(X^*) \subseteq \{1, \dots, i^1\}$, $|N(X^*)| = |X^*| \le k$.

Proof. Property (i) is proved by noting that $\sum_{i\in N} \frac{c_i^*-\underline{c}_i}{d_i} = 1 \le \Gamma$, and the cost of the solution (X^*, c^*, x^*) is equal to $D_{\min}$. Property (ii) is proved by a contradiction method. Consider an optimal solution (X^*, c^*, x^*) and assume that $\Delta^* > D_{\min}$. Denote by x^0 the 0–1 vector with single 1 in position i^0. Set $X^0 = \{x^0\}$ and observe that the cost of a solution (X^0, c, x^0) does not exceed $D_{\min}$ for any feasible cost vector c. Therefore, (X^*, c^*, x^*) is not optimal, and the assumption $\Delta^* > D_{\min}$ is wrong. For property (iii), if $\Delta^* < \underline{c}_i$ and (X^*, c^*, x^*) is an optimal solution, then it is easy to see that $x_i^* = 0$, since otherwise $\Delta^* \ge \underline{c}_i$ by the definition of Δ^*. As a consequence, a solution $(X^* \setminus \{x\}, c^*, x^*)$, where $x_i = 1$, is also optimal for 1-Min-Max-Min. Property (iv) directly follows from property (iii), since if $\Delta^* < D_{\min}$, then $\Delta^* < \underline{c}_i$ for any $i > i^1$ by the definition of i^1. □

Property (iv) of Lemma 1 implies that there exists an optimal solution of 1-Min-Max-Min in which no representative $i > i^1$ is selected as an alternative. It is convenient to denote an optimal solution of the problem 1-Max-Min(X) as an ordered pair $(c^{(X)}, x^{(X)})$, where $c^{(X)} \in U$ and $x^{(X)} \in X$, and denote by $\Delta^{(X)}$ its optimal objective function value, which is calculated as $\Delta^{(X)} = \max_{c\in U} \min_{x\in X} \sum_{i\in N} c_i x_i$. For solving the original problem 1-Min-Max-Min, we are interested in the sets $X \subseteq X^{(1)}$ such that properties (ii), (iii) and (iv) of the problem 1-Max-Min(X) are satisfied when X and $\Delta^{(X)}$ replace X^* and Δ^* in Lemma 1.

Lemma 2. *If properties (ii)-(iv) are satisfied for $X^* = X$ and $\Delta^* = \Delta^{(X)}$, then there exists an optimal solution $(c^{(X)}, x^{(X)})$ of the problem* 1-Max-Min *that satisfies the following properties.*
(v) For each $i \in N\backslash N(X)$, if $\Delta^{(X)} < \underline{c}_i$, then $c_i^{(X)} = \underline{c}_i$.
(vi) For each $i \in N(X)$, relation $\underline{c}_i \le c_i^{(X)} = \Delta^{(X)} \le \underline{c}_i + d_i$ is satisfied, and optimal vector $x^{(X)}$ is any vector from X.
(vii) If $\Delta^{(X)} < D_{\min}$, then $\sum_{i\in N(X)} \frac{\Delta^{(X)}-\underline{c}_i}{d_i} = \Gamma$, and equivalently, $\Delta^{(X)} = (\Gamma + \sum_{i\in N(X)} \frac{\underline{c}_i}{d_i})/\sum_{i\in N(X)} \frac{1}{d_i}$.

Proof. Property (v) is proved by noting that if $i \in N\backslash N(X)$ then re-setting $c_i^{(X)} := \underline{c}_i$ keeps the new solution optimal. Assume that properties (i)-(iv) are satisfied (for X and $\Delta^{(X)}$) and the first statement of property (vi) is not. This latter assumption can also be re-phrased such that the minimal and maximal values among $c_i^{(X)}$, $i \in N(X)$, are different. Denote $c_a^{(X)} = \min_{i\in N(X)} c_i^{(X)}$ and $c_b^{(X)} = \max_{i\in N(X)} c_i^{(X)}$. If the first statement of property (vi) is incorrect, then

$$\max\{\underline{c}_a, \underline{c}_b\} \le \Delta^{(X)} = c_a^{(X)} < c_b^{(X)} \le D_{\min} \le \min\{\underline{c}_a + d_a, \underline{c}_b + d_b\}.$$

Re-setting (increasing) $c_a^{(X)} := c_a^{(X)} + \frac{c_b^{(X)}-c_a^{(X)}}{d_a+d_b} \cdot d_a$, and re-setting (decreasing) $c_b^{(X)} := c_b^{(X)} - \frac{c_b^{(X)}-c_a^{(X)}}{d_a+d_b} \cdot d_b$ leads to

$$\max\{\underline{c}_a, \underline{c}_b\} \le c_a^{(X)} = c_b^{(X)} \le D_{\min} \le \min\{\underline{c}_a + d_a, \underline{c}_b + d_b\}.$$

This also does not change $\frac{c_a^{(X)}-\underline{c}_a}{d_a} + \frac{c_b^{(X)}-\underline{c}_b}{d_b}$, and consequently, the left-hand side $\sum_{i\in N} \frac{c_i^{(X)}-\underline{c}_i}{d_i}$ of the budget constraint. By repeating this re-setting argument at most $|N(X)| - 1$ times, we come to an optimal solution of 1-Max-Min(X) with $\Delta^{(X)} = \min_{i\in N(X)} c_i^{(X)} = \max_{i\in N(X)} c_i^{(X)}$, as it is required in the first statement of property (vi). Since by this statement $\sum_{i\in N} c_i x_i = \Delta^{(X)}$ for any $x \in X$, then any vector from X is optimal, as it is required in the second statement of this property.

To prove property (vii), consider an optimal solution of 1-Max-Min(X) satisfying properties (ii)-(vi) such that $c_i^{(X)} = \Delta^{(X)}$, $i \in N(X)$, $c_i^{(X)} = \underline{c}_i$, $i \in N\backslash N(X)$, and assume that $\sum_{i\in N(X)} \frac{\Delta^{(X)}-\underline{c}_i}{d_i} := \Gamma^0 < \Gamma$. Observe that $\sum_{i\in N(X)} \left(\Delta^{(X)} + \frac{\Gamma-\Gamma^0}{\sum_{i\in N(X)} \frac{1}{d_i}} - \underline{c}_i\right) / d_i = \Gamma$. Therefore, increasing the costs to become $c_i^{(X)} := \min\left\{\underline{c}_i + d_i, \Delta^{(X)} + \frac{\Gamma-\Gamma^0}{\sum_{i\in N(X)} \frac{1}{d_i}}\right\}$ for $i \in N(X)$ either leads to an optimal solution satisfying properties (ii)-(vi) and $\sum_{i\in N(X)} \frac{c_i^{(X)}-\underline{c}_i}{d_i} = \Gamma$, or to the case $\Delta^{(X)} = D_{\min}$, already described in property (i) for $X^* = X$. □

Introduce a collection $\mathcal{X}^{(\Delta)}$ of sets $X \subseteq X^{(1)}$ such that $N(X) \subseteq \{1, \ldots, i^1\}$, $|N(X)| = |X| \le k$, and $i \in N\backslash N(X)$ if $\Delta < \underline{c}_i$.

Lemma 3. *The following properties of the problem* 1-Min-Max-Min *are satisfied.*
(viii) If $\Delta^* < D_{\min}$, then set X^* satisfying Property (vii) is a maximizer of the function $\max_{X\in\mathcal{X}^{(\Delta^*)}}\left\{\sum_{i\in N(X)} \frac{\Delta^*-\underline{c}_i}{d_i}\right\}$:

$$\max_{X\in\mathcal{X}^{(\Delta^*)}}\left\{\sum_{i\in N(X)} \frac{\Delta^*-\underline{c}_i}{d_i}\right\} = \sum_{i\in N(X^*)} \frac{\Delta^*-\underline{c}_i}{d_i} = \Gamma.$$

(ix) Let $\underline{c}_1 \le \Delta < D_{\min}$. Relation $\max_{X\in\mathcal{X}^{(\Delta)}}\left\{\sum_{i\in N(X)} \frac{\Delta-\underline{c}_i}{d_i}\right\} \le \Gamma$ is satisfied if and only if $\Delta \le \Delta^*$.

Proof. Assume that property ($viii$) is wrong, *i.e.*, $\Delta^* < D_{\min}$ and there exists $X' \in \mathcal{X}^{(\Delta^*)} \subseteq X^{(1)}$ such that

$$\sum_{i\in N(X')} \frac{\Delta^*-\underline{c}_i}{d_i} > \sum_{i\in N(X^*)} \frac{\Delta^*-\underline{c}_i}{d_i} = \Gamma. \qquad (2)$$

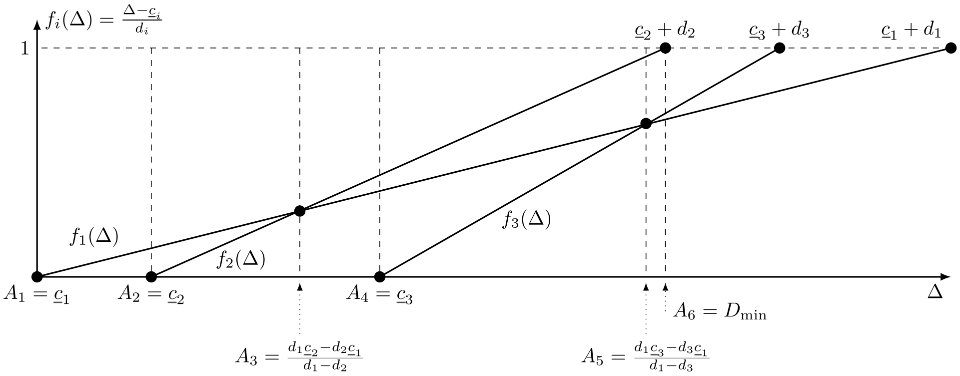

Fig. 2. Functions $f_i(\Delta)$, $\underline{c}_i \le \Delta \le \underline{c}_i + d_i$, $i \in \{1, \ldots, i^1\}$, for the example in Fig. 1. If $k = 2$, then $N(X^{[A_1,A_2]}) = \{1\}$, $N(X^{[A_2,A_3]}) = N(X^{[A_3,A_4]}) = N(X^{[A_4,A_5]}) = \{1, 2\}$, $N(X^{[A_5,A_6]}) = \{2, 3\}$. Vertical lines correspond to the points $A_1, \ldots, A_u$. Horizontal lines correspond to the values 0 and 1 of the functions $f_i(\Delta)$.

Consider problem 1-Max-Min(X') with the fixed set X'. Clearly, $\sum_{i\in N(X')} \frac{\Delta(X')-\underline{c}_i}{d_i} \le \Gamma$ and $\Delta^* \le \Delta(X')$ by the definition of Δ^*. If $\Delta^* = \Delta(X')$, then $\Gamma \ge \sum_{i\in N(X')} \frac{\Delta(X')-\underline{c}_i}{d_i} = \sum_{i\in N(X')} \frac{\Delta^*-\underline{c}_i}{d_i}$, and (2) is wrong. If $\Delta^* < \Delta(X')$, then, based on (2) and on the fact that $\sum_{i\in N(X')} \frac{\Delta-\underline{c}_i}{d_i}$ is increasing in Δ, we have to obtain

$$\Gamma = \sum_{i\in N(X^*)} \frac{\Delta^* - \underline{c}_i}{d_i} < \sum_{i\in N(X')} \frac{\Delta^* - \underline{c}_i}{d_i} < \sum_{i\in N(X')} \frac{\Delta(X') - \underline{c}_i}{d_i} \le \Gamma,$$

which is a contradiction. Thus, $\max_{X\in\mathcal{X}^{(\Delta^*)}}\left\{\sum_{i\in N(X)} \frac{\Delta^*-\underline{c}_i}{d_i}\right\} = \sum_{i\in N(X^*)} \frac{\Delta^*-\underline{c}_i}{d_i} = \Gamma$.

Note that $\mathcal{X}^{(\Delta')} \subseteq \mathcal{X}^{(\Delta'')}$ if $\underline{c}_1 \le \Delta' \le \Delta'' < D_{\min}$. Indeed, it can be seen from Fig. 1 that if $X \in \mathcal{X}^{(\Delta^*)}$ (X corresponds to a subset of lines crossing the vertical line $\Delta = \Delta^*$), then $X \in \mathcal{X}^{(\Delta)}$ for any Δ such that $\Delta^* \le \Delta < D_{\min}$. In order to prove part "only if" of property (ix), assume that $\max_{X\in\mathcal{X}^{(\Delta)}}\left\{\sum_{i\in N(X)} \frac{\Delta-\underline{c}_i}{d_i}\right\} \le \Gamma$. Furthermore, assume that the part "only if" is wrong, that is, $\Delta^* < \Delta$. The latter inequality implies $X^* \in \mathcal{X}^{(\Delta)}$ and

$$\sum_{i\in N(X^*)} \frac{\Delta - \underline{c}_i}{d_i} \le \max_{X\in\mathcal{X}^{(\Delta)}}\left\{\sum_{i\in N(X)} \frac{\Delta - \underline{c}_i}{d_i}\right\} \le \Gamma = \sum_{i\in N(X^*)} \frac{\Delta^* - \underline{c}_i}{d_i}.$$

Since function $\sum_{i\in N(X)} \frac{\Delta-\underline{c}_i}{d_i}$ is increasing in Δ for any fixed X, the above relations imply $\Delta \le \Delta^*$. This contradiction proves the part "only if". For the part "if", consider value Δ such that $\underline{c}_1 \le \Delta \le \Delta^* < D_{\min}$. Relation $\Delta \le \Delta^*$ implies $\mathcal{X}^{(\Delta)} \subseteq \mathcal{X}^{(\Delta^*)}$ and $\max_{X\in\mathcal{X}^{(\Delta)}}\left\{\sum_{i\in N(X)} \frac{\Delta-\underline{c}_i}{d_i}\right\} \le \max_{X\in\mathcal{X}^{(\Delta)}}\left\{\sum_{i\in N(X)} \frac{\Delta^*-\underline{c}_i}{d_i}\right\} \le \max_{X\in\mathcal{X}^{(\Delta^*)}}\left\{\sum_{i\in N(X)} \frac{\Delta^*-\underline{c}_i}{d_i}\right\} = \Gamma$. □

Our further discussion is supported by Fig. 2.

Consider linear functions $f_i(\Delta) = \frac{\Delta-\underline{c}_i}{d_i}$ defined for $\underline{c}_i \le \Delta \le \underline{c}_i + d_i$, and therefore, having values $0 \le f_i(\Delta) \le 1$, $i \in \{1, \ldots, i^1\}$. Observe that if these functions do not intersect for any $\Delta \in (A, B)$, $\underline{c}_1 \le A < B < D_{\min}$ and $A \le \Delta^* \le B$, then the set X^* in property ($viii$) is independent of Δ^* and it is uniquely determined by the at most k upper lines in the interval (A, B). Denote this set as $X^{[A,B]}$. By properties ($viii$) and (ix), if $\sum_{i\in N(X^{[A,B]})} \frac{A-\underline{c}_i}{d_i} \le \Gamma \le \sum_{i\in N(X^{[A,B]})} \frac{B-\underline{c}_i}{d_i}$, then $X^* = X^{[A,B]}$, and by property (vii),

$$\Delta^* = \frac{\Gamma + \sum_{i\in N(X^{[A,B]})} \frac{\underline{c}_i}{d_i}}{\sum_{i\in N(X^{[A,B]})} \frac{1}{d_i}} := \Delta^{[A,B]}.$$

Let $A_1, \ldots, A_u$ be a strictly increasing sequence of all distinct Δ-points from the set of points consisting of the point $D_{\min}$, points $\underline{c}_i$, $i \in \{1, \ldots, i^1\}$, and Δ-points of intersection of the lines $f_i(\Delta)$, $i \in \{1, \ldots, i^1\}$, within the rectangle with the bottom left corner $(\underline{c}_1, 0)$ and the top right corner $(D_{\min}, 1)$. We have $u \le O(n^2)$. Since intervals $[A_v, A_{v+1}]$, $v \in \{1, \ldots, u-1\}$, cover the range $[\underline{c}_1, D_{\min})$ of Δ^* in the case $\Delta^* < D_{\min}$, and linear functions $f_i(\Delta)$, $i \in \{1, \ldots, i^1\}$, do not intersect for any $\Delta \in (A_v, A_{v+1})$, we know that

$$\Delta^* = \min\left\{D_{\min}, \min_{1\le v\le u-1}\left\{\Delta^{[A_v,A_{v+1}]} \;\middle|\; \Gamma < \sum_{i\in N(X^{[A_v,A_{v+1}]})} \frac{A_{v+1} - \underline{c}_i}{d_i}\right\}\right\}. \tag{3}$$

If $\sum_{i\in N(X^{[A_{u-1},A_u]})} \frac{A_u-\underline{c}_i}{d_i} \le \Gamma$, then $\Delta^* = D_{\min}$ and solution (X^*, c^*, x^*) in property (i) of Lemma 1 is optimal. Otherwise, Δ^* is attained at the smallest index v for which the condition $\Gamma < \sum_{i\in N(X^{[A_v,A_{v+1}]})} \frac{A_{v+1}-\underline{c}_i}{d_i}$ is satisfied. If this index is v^*, then a solution $(X^{[A_{v^*},A_{v^*+1}]}, c^*, x^*)$ is optimal, in which $c_i^* = \Delta^{[A_{v^*},A_{v^*+1}]}$, $i \in N(X^{[A_{v^*},A_{v^*+1}]})$, $c_i^* = \underline{c}_i$, $i \in N\backslash N(X^{[A_{v^*},A_{v^*+1}]})$ and x^* is any vector from $X^{[A_{v^*},A_{v^*+1}]}$. We further describe an approach to find v^*.

Introduce mid-points $M_v = \frac{A_v+A_{v+1}}{2}$ of the intervals $[A_v, A_{v+1}]$, $v \in \{1,\dots,u-1\}$. Denote by N_v the set of functions $f_i(\Delta)$ which are present in the Δ-interval $[A_v, A_{v+1}]$, *i.e.*, $N_v = \{i \in \{1,\dots,i^1\} \mid 0 < f_i(M_v) < 1\}$, $v \in \{1,\dots,u-1\}$. Define a "budget constraint function"

$$T(\Delta) = \sum_{i\in N(X^{[A_v,A_{v+1}]})} \frac{\Delta-\underline{c}_i}{d_i} = \max_{S\subseteq N_v, |S|\le k}\left\{\sum_{i\in S}\frac{\Delta-\underline{c}_i}{d_i}\right\},$$

if $\Delta \in (A_v, A_{v+1}]$, $v \in \{1,\dots,u-1\}$, and $T(A_1) = 0$.

It is possible to calculate $N(X^{[A_v,A_{v+1}]})$, and consequently $T(\Delta)$, using the mid-points since the functions do not intersect in the interval (A_v, A_{v+1}). Note that, since $\Delta \in (A_v, A_{v+1}]$ in the above definition, the unique set $N(X^{[A_v,A_{v+1}]})$ out of the two candidate sets $N(X^{[A_{v+1},A_{v+2}]})$ and $N(X^{[A_v,A_{v+1}]})$ is used to calculate $T(\Delta)$ at the intersection point A_{v+1}, $v \in \{1,\dots,u-1\}$.

Lemma 4. *$T(\Delta)$ is increasing piece-wise linear function.*

Proof. The function $T(\Delta)$ is linear increasing for Δ in the same interval $[A_v, A_{v+1}]$, $v \in \{1,\dots,u-1\}$. Furthermore, it is continuous because $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_{u-1}$. Therefore, it is increasing piece-wise linear. □

The following lemma demonstrates that a bisection search can be used to find index v^* at which inner minimum in (3) is attained if it exists.

Lemma 5. *If $T(\Delta) \le \Gamma$ for $\underline{c}_1 \le \Delta \le D_{\min}$, then $\Delta^* \ge \Delta$, else $\Delta^* < \Delta$. In particular, if $T(A_v) \le \Gamma$ for some v, $1 \le v \le u$, then $\Delta^* \ge A_v$, else $\Delta^* < A_v$.*

Proof. This lemma is a re-statement of property (ix) in Lemma 3 in terminology of the function $T(\Delta)$. □

Our discussion justifies the following algorithm for the problem 1-Min-Max-Min. Recall that $1 \le \Gamma \le k$.

Algorithm A.

Step 1 Re-number representatives such that $\underline{c}_1 \le \cdots \le \underline{c}_n$. Calculate indices i^0 and i^1, points $A_1,\dots,A_u$ and value $T(A_u)$. If $T(A_u) \le \Gamma$, then output optimal solution (X^*, c^*, x^*) with the cost $D_{\min} = \underline{c}_{i^0} + d_{i^0} = A_u$ as in property (i) of Lemma 1 and stop. If $T(A_u) > \Gamma$, then define initial lower and upper bounds on the index v^* as $L = 1$ and $U = u$ and perform a bisection search over indices $v \in \{1,\dots,u\}$ in Step 2.

Step 2 Calculate $v = \left\lfloor \frac{L+U}{2} \right\rfloor$. If $v = L$, then perform Step 3, else perform the following computations. Calculate set $N(X^{[A_v,A_{v+1}]})$ and value $T(M_v)$. If $T(M_v) \le \Gamma$, then re-set $L := v$ and repeat Step 2. If $T(M_v) > \Gamma$, then re-set $U := v$ and repeat Step 2.

Step 3 Find optimal solution (X^*, c^*, x^*) with cost value Δ^* such that $X^* = X^{[A_v,A_{v+1}]}$, $c_i^* = \Delta^*$, $i \in N(X^*)$, $c_i^* = \underline{c}_i$, $i \in N\backslash N(X^*)$, $\Delta^* = (\Gamma + \sum_{i\in N(X^*)} \frac{\underline{c}_i}{d_i})/\sum_{i\in N(X^*)} \frac{1}{d_i}$, x^* is any vector from X^*, and stop.

Step 1 requires $O(n^2 \log n)$ time to calculate $O(n^2)$ intersection points of n lines and sort the distinct intersection points A_v in the increasing order. The number of iterations of the bisection search in Step 2 is at most $O(\log n)$, and each iteration needs $O(n)$ time to find at most $k \le n$ largest values of n linear functions at the mid-point M_v and to further calculate $T(M_v)$. Step 3 needs $O(n)$ time to retrieve an optimal solution associated with the last verified point M_v. Thus, the overall running time of Algorithm A is $O(n^2 \log n)$.

Comment 1. *For the case of $T(A_u) \le \Gamma$ in Step 1, extending $X^* = \{x^*\}$ by arbitrary $k-1$ vectors from $X^{(1)}$ does not make the optimal solution (X^*, c^*, x^*) worse.*

4. Conclusion

We have studied a robust cost minimizing single representative selection problem with budgeted uncertainty, denoted as 1-Min-Max-Min. This problem is a two-stage decision problem. A set of at most k alternative representatives should be determined in the first stage and a single alternative representative should be selected in the second stage, after the representatives worst costs are realized. The deterministic version of this problem is trivial — selecting a representative with the minimum cost is optimal.

The problem 1-Min-Max-Min is a special case of the problem studied by Chassein and Goerigk [15], in which p representatives have to be selected. An $O(n^4 \log n)$ time algorithm was presented in [15] for the special case of the more general problem, in which the number of alternative solutions is $k = 2$. We proposed an $O(n^2 \log n)$ algorithm for the problem 1-Min-Max-Min, thus showing that the case of the more general problem with arbitrary k and $p = 1$ is polynomially solvable. It is interesting to know whether the $O(n^2 \log n)$ time complexity can be reduced, whether the solution technique for $p = 1$ can be extended for $p \geq 2$, and what is the complexity status of the more general problem with variable k and $p \geq 2$.

Data availability

No data was used for the research described in the article.

References

[1] A. Agra, M.C. Santos, D. Nace, M. Poss, A dynamic programming approach for a class of robust optimization problems, SIAM J. Optim. 26 (3) (2016) 1799–1823.
[2] S. Albers, M. Janke, Online makespan minimization with budgeted uncertainty, in: A. Lubiw, M. Salavatipour, M. He (Eds.), Algorithms and Data Structures, in: Lecture Notes in Computer Science, Vol. 12808, Springer, Cham, 2021, pp. 43–56.
[3] Z. Ales, T.S. Nguyen, M. Poss, Minimizing the weighted sum of completion times under processing time uncertainty, Electron. Notes Discrete Math. 64 (2018) 15–24.
[4] A. Alves Pessoa, L. Di Puglia Pugliese, F. Guerriero, M. Poss, Robust constrained shortest path problems under budgeted uncertainty, Networks 66 (2) (2015) 98–111.
[5] J. Ayoub, M. Poss, Decomposition for adjustable robust linear optimization subject to uncertainty polytope, Comput. Manag. Sci. 13 (2) (2016) 219–239.
[6] P. Bendotti, P. Chrétienne, P. Fouilhoux, A. Pass-Lanneau, Dominance-based linear formulation for the anchor-robust project scheduling problem, European J. Oper. Res. 295 (1) (2021) 22–33.
[7] D. Bertsimas, M. Sim, Robust discrete optimization and network flows, Math. Program. 98 (1–3) (2003) 49–71.
[8] D. Bertsimas, M. Sim, The price of robustness, Oper. Res. 52 (1) (2004) 35–53.
[9] M. Bold, M. Goerigk, A compact reformulation of the two-stage robust resource-constrained project scheduling problem, Comput. Oper. Res. 130 (2021) 105232.
[10] M. Bold, M. Goerigk, A faster exact method for solving the robust multi-mode resource-constrained project scheduling problem, Oper. Res. Lett. 50 (5) (2022) 581–587.
[11] M. Bougeret, K. Jansen, M. Poss, L. Rohwedder, Approximation results for makespan minimization with budgeted uncertainty, Theory Comput. Syst. 65 (6) (2021) 903–915.
[12] M. Bougeret, A.A. Pessoa, M. Poss, Robust scheduling with budgeted uncertainty, Discrete Appl. Math. 261 (2019) 93–107.
[13] M. Bougeret, A. Pessoa, M. Poss, Single machine robust scheduling with budgeted uncertainty, Oper. Res. Lett. 51 (2) (2023) 137–141.
[14] M. Bruni, L. Di Puglia Pugliese, P. Beraldi, F. Guerriero, A computational study of exact approaches for the adjustable robust resource-constrained project scheduling problem, Comput. Oper. Res. 99 (2018) 178–190.
[15] A. Chassein, M. Goerigk, On the complexity of min–max–min robustness with two alternatives and budgeted uncertainty, Discrete Appl. Math. 296 (2021) 141–163.
[16] A. Chassein, M. Goerigk, A. Kasperski, P. Zieliński, On recoverable and two-stage robust selection problems with budgeted uncertainty, European J. Oper. Res. 265 (2) (2018) 423–436.
[17] A. Chassein, M. Goerigk, J. Kurtz, M. Poss, Faster algorithms for min–max–min robustness for combinatorial problems with budgeted uncertainty, European J. Oper. Res. 279 (2) (2019) 308–319.
[18] L. Di Puglia Pugliese, F. Guerriero, M. Poss, The resource constrained shortest path problem with uncertain data: A robust formulation and optimal solution approach, Comput. Oper. Res. 107 (2019) 140–155.
[19] M. Goerigk, M. Hartisch, Combinatorial optimization problems with balanced regret, Discrete Appl. Math. 328 (2023) 40–59.
[20] M. Goerigk, J. Kurtz, M. Poss, Min–max–min robustness for combinatorial problems with discrete budgeted uncertainty, Discrete Appl. Math. 285 (2020) 707–725.
[21] M. Goerigk, S. Lendl, L. Wulf, Recoverable robust representatives selection problems with discrete budgeted uncertainty, European J. Oper. Res. 303 (2) (2022) 567–580.
[22] M. Goerigk, S. Lendl, L. Wulf, Two-stage robust optimization problems with two-stage uncertainty, European J. Oper. Res. 302 (1) (2022) 62–78.
[23] I. Hamaz, L. Houssin, S. Cafieri, A robust basic cyclic scheduling problem, EURO J. Comput. Optim. 6 (3) (2018) 291–313.
[24] M. Levorato, R. Figueiredo, Y. Frota, Exact solutions for the two-machine robust flow shop with budgeted uncertainty, European J. Oper. Res. 300 (1) (2022) 46–57.
[25] M. Levorato, D. Sotelo, R. Figueiredo, Y. Frota, Robust permutation flow shop total weighted completion time problem: Solution and application to the oil and gas industry, Comput. Oper. Res. 151 (2023) 106117.
[26] S. Mattia, M. Poss, A comparison of different routing schemes for the robust network loading problem: polyhedral results and computation, Comput. Optim. Appl. 69 (3) (2018) 753–800.
[27] M. Poss, Robust combinatorial optimization with variable budgeted uncertainty, 4OR 11 (1) (2013) 75–92.
[28] M. Silva, M. Poss, N. Maculan, Solution algorithms for minimizing the total tardiness with budgeted processing time uncertainty, European J. Oper. Res. 283 (1) (2020) 70–82.