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ON THE REGULARIZATION OF VECTOR INTEGER QUADRATIC PROGRAMMING PROBLEMS

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For a vector integer quadratic programming problem, a regularizing operator is proposed that acts on a vector criterion and transforms a possibly unstable initial problem into a series of perturbed stable problems with the same Pareto set. The technique of ε-regularization is developed that allows replacing the considered problem by perturbed ε-stable problems.

Keywords: *multicriteriality, integer quadratic programming, Pareto set, stability, stabilization, regularization, ε-stabilization, ε-regularization.*

INTRODUCTION

As is well known [1–3], scalar (one-criterion) linear discrete optimization problems are always stable. The existence of unstable (or ill-posed in the sense of Hadamard) vector discrete optimization problems [1, 2, 4–6] naturally leads to the necessity of creation of a regularizing operator that is a concrete kind of perturbations of initial data of a discrete problem in order that, as in the case of a linear programming problem [7, 8], to replace a possibly unstable problem by a series of perturbed stable ones. The first result in this direction is obtained in [9] in which, based on the theory of cones of promising directions [1], a regularization is proposed with respect to a vector criterion and constraints of the vector integer linear programming (ILP) problem of searching for the Pareto set on a bounded set of admissible alternatives. In this case, the regularization with respect to a vector criterion is based on a scalar perturbation parameter and implies the replacement of a vector ILP problem by a perturbed problem with the Slater set contained in the Pareto set of the original problem. Later on, this result is generalized in [10, 11] in which not a scalar but a vector perturbation parameter of the initial data of a problem is considered that allows the creation of a regularizing operator that transforms a possibly unstable vector ILP problem into a series of not only stable but also equivalent problems, i.e., problems with the original Pareto set. A discussion and a comparison of the results of [9, 10] are presented in [2, pp.155–163].

In this article, the methodology developed in [10] is applied to the vector integer quadratic programming (IQP) problem consisting of searching for Pareto sets. An expedient of regularization and ε-regularization of such problems is proposed.

We note that an expedient is described in [12, 13] for regularization and ε-regularization of vector integer linear and quadratic programming problems with the lexicographic optimality principle.

1. BASIC DEFINITIONS AND LEMMAS

Let us consider a vector (m-criterion) IQP problem of the form

$$Z^m(A, b): \min\{f(x, A, b): x \in X\}, \quad m \ge 1,$$

where $f(x, A, b) = (f_1(x, A_1, b_1), f_2(x, A_2, b_2), \dots, f_m(x, A_m, b_m))$, $f_i: \mathbf{R}^{n\times n} \times \mathbf{R}^n \to \mathbf{R}$, are quadratic functions

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(partial criterions), i.e., we have

$$f_i(x, A_i, b_i) = \langle A_i x, x\rangle + \langle b_i^{\mathrm{T}}, x\rangle, \ i \in N_m = \{1, 2, \dots, m\}.$$

Here, $X \subset \mathbf{Z}^n$ is a given finite set of integral points (column vectors) in $\mathbf{R}^n$, $n \geq 1$, $A = (a_{ijk}) \in \mathbf{R}^{m\times n\times n}$ is a three-index matrix of dimension $m \times n \times n$, $A_i \in \mathbf{R}^{n\times n}$, $i \in N_m$, are its two-dimensional sections, and $b \in \mathbf{R}^{m\times n}$ is a matrix with rows $b_i \in \mathbf{R}^n$, $i \in N_m$, $|X| > 1$.

We consider the problem $Z^m(A, b)$ as the problem of searching for the Pareto set (the set of efficient points) $P^m(A, b)$ that is specified by the formula

$$P^m(A, b) = \{x \in X : \pi(x, A, b) = \varnothing\},$$

where

$$\pi(x, A, b) = \{x' \in X : q(x, x', A, b) \geq \mathbf{0} \ \& \ q(x, x', A, b) \neq \mathbf{0}\},$$

$$q(x, x', A, b) = (q_1(x, x', A_1, b_1), q_2(x, x', A_2, b_2), \dots, q_m(x, x', A_m, b_m)),$$

$$q_i(x, x', A_i, b_i) = f_i(x, A_i, b_i) - f_i(x', A_i, b_i), \ i \in N_m,$$

$$\mathbf{0} = (0, 0, \dots, 0) \in \mathbf{R}^m.$$

It is obvious that $P^1(A, b)$ is the set of optimal solutions of a scalar quadratic problem with a large application domain. Among such problems are quadratic assignment problems and many well-known optimization problems on graphs that play an important role in designing electronic hardware, in particular, the problems of placement, partitioning, layout, packing, etc. [14].

For any pair $(A, b) \in \mathbf{R}^{m\times n\times n} \times \mathbf{R}^{m\times n}$, the following inclusion is takes place:

$$P^m(A, b) \subseteq \mathrm{Sl}^m(A, b). \tag{1}$$

Here, $\mathrm{Sl}^m(A, b)$ is the Slater set (the set of weakly efficient points),

$$\mathrm{Sl}^m(A, b) = \{x \in X : \sigma(x, A, b) = \varnothing\},$$

where

$$\sigma(x, A, b) = \{x' \in X : \forall i \in N_n \ (q_i(x, x', A_i, b_i) > 0)\}.$$

In what follows, we will use the denotation $\overline{P^m}(A, b) = X \setminus P^m(A, b)$.

The lemma formulated below directly follows from the above denotations.

LEMMA 1. If, for any vector $x \in \overline{P^m}(A, b)$, the inclusion $\pi(x, A, b) \subseteq \sigma(x, C, d)$ is true, then we have $\mathrm{Sl}^m(C, d) \subseteq P^m(A, b)$.

For any natural number k, we specify norms l_∞ and l_1 in the space $\mathbf{R}^k$ that are determined, respectively, by the formulas

$$\|y\| = \max\{|y_i| : i \in N_k\}, \ \|y\|_* = \sum_{i=1}^{k} |y_i|.$$

By the norm of a matrix we understand the norm of the vector composed of its components.

LEMMA 2. For any $i \in N_m$ and $x, x' \in X$, the following inequality is true:

$$q_i(x, x', A_i, b_i) \leq (\|A\| + \|b\|)\|x - x'\|_*^2.$$

In fact, using the obvious inequalities

$$\forall B \in \mathbf{R}^{n\times n} \ \ \forall x \in \mathbf{R}^n \ \ (|\langle Bx, x\rangle| \leq \|B\| \cdot \|x\|_*^2),$$

$$\forall c \in \mathbf{R}^n \ \ \forall x \in \mathbf{R}^n \ \ (|\langle c, x\rangle| \leq \|c\| \cdot \|x\|_*),$$

$$\forall x \in \mathbf{Z}^n \ \ \forall x' \in \mathbf{Z}^n \ \ (\|x - x'\|_* \leq \|x - x'\|_*^2),$$

we easily derive the inequalities

$$q_i(x,x',A_i,b_i) \le |q_i(x,x',A_i,b_i)| \le |\langle A_i(x-x'),x-x'\rangle| + |\langle b_i,x-x'\rangle|$$
$$\le \|A_i\|\cdot\|x-x'\|_*^2 + \|b_i\|\cdot\|x-x'\|_* \le (\|A\|+\|b\|)\|x-x'\|_*^2.$$

We call the problem $Z^m(A,b)$ stable (T_3-stable with respect to a vector criterion in the terminology of [2, 4–6]) if there is a number $\varepsilon > 0$ such that we have

$$\forall\,(A',b') \in \Xi(\varepsilon)\ \ (P^m(A+A',b+b') \subseteq P^m(A,b)), \tag{2}$$

where the set of perturbing pairs $\Xi(\varepsilon)$ is specified by the equality

$$\Xi(\varepsilon) = \{(A',b') \in \mathbf{R}^{m\times n\times n} \times \mathbf{R}^{m\times n} : \max\{\|A'\|,\|b'\|\} < \varepsilon\}.$$

As is obvious, the problem $Z^m(A,b)$ is stable if we have $\overline{P^m}(A,b)=\varnothing$. In what follows, we exclude this case from consideration and call the problem for which we have $\overline{P^m}(A,b)\ne\varnothing$ nontrivial.

It is obvious that, in the particular case when $m=1$, the Pareto set coincides with the Slater set and, hence, according to [5], the scalar IQP problem is stable. In this connection, we assume that the number of partial criterions $m \ge 2$.

In what follows, the following well-known result [5] will be required.

LEMMA 3. There is a number $\varepsilon > 0$ such that, for any pair $(A',b') \in \Xi(\varepsilon)$, the following inclusion is fulfilled:

$$\mathrm{Sl}^m(A+A',b+b') \subseteq \mathrm{Sl}^m(A,b).$$

In other words, Lemma 3 asserts that a vector IQP problem of finding the Slater set is always stable.

Let $(C,d) \in \mathbf{R}^{m\times n\times n} \times \mathbf{R}^{m\times n}$.

We call a mapping $\varphi: \mathbf{R}^{m\times n\times n} \times \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n\times n} \times \mathbf{R}^{m\times n}$ specified by the rule

$$\forall (A,b) \in \mathbf{R}^{m\times n\times n} \times \mathbf{R}^{m\times n}\ \ (\varphi(A,b) = (A+C,b+d))$$

a (C,d)-operator. Thereby, a (C,d)-operator transforms any problem $Z^m(A,b)$ into the problem $Z^m(A+C,b+d)$.

We say that a (C,d)-operator stabilizes a problem $Z^m(A,b)$ if there is a number $\varepsilon > 0$ such that the following formula is true:

$$\forall (A',b') \in \Xi(\varepsilon)\ \ (P^m(A+C+A',b+d+b') \subseteq P^m(A,b)). \tag{3}$$

We will use the denotation $\mathbf{R}_> = \{u \in \mathbf{R} : u > 0\}$.

For any vector $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_m) \in \mathbf{R}_>^m$ and any number $\tau > 0$, we consider a three-index matrix $C(A,\lambda,\tau) \in \mathbf{R}^{m\times n\times n}$ with identical sections

$$C_i = \tau \sum_{k=1}^{m} \lambda_k A_k,\ \ i \in N_m,$$

and a matrix $d(b,\lambda,\tau) \in \mathbf{R}^{m\times n}$ with identical rows

$$d_i = \tau \sum_{k=1}^{m} \lambda_k b_k,\ \ i \in N_m.$$

Then the following equalities are obvious:

$$q_i(x,x',C_i,d_i) = \tau \sum_{k=1}^{m} \lambda_k q_k(x,x',A_k,b_k),\ \ i \in N_m. \tag{4}$$

These equalities imply two equivalent obvious statements that are formulated in the form of the lemma presented below.

LEMMA 4. Let $x, x' \in X$, let $\lambda \in \mathbf{R}^m_>$, and let $\tau > 0$. If $q(x, x', A, b) = \mathbf{0}$, then we have $q(x, x', A + C(A, \lambda, \tau), b + d(b, \lambda, \tau)) = \mathbf{0}$. If $q(x, x', A + C(A, \lambda, \tau), b + d(b, \lambda, \tau)) \neq \mathbf{0}$, then we obtain $q(x, x', A, b) \neq \mathbf{0}$.

Let $A, C \in \mathbf{R}^{m \times n \times n}$, and let $b, d \in \mathbf{R}^{m \times n}$. We call problems $Z^m(A, b)$ and $Z^m(C, d)$ equivalent if we have $P^m(A, b) = P^m(C, d)$. In this case, we write $Z^m(A, b) \sim Z^m(C, d)$. We say that a (C, d)-operator regularizes a problem $Z^m(A, b)$ if we have $Z^m(A, b) \sim Z^m(A + C, b + d)$ and the problem $Z^m(A + C, b + d)$ is stable. It is obvious that the (C, d)-operator regularizes the problem $Z^m(A, b)$ if it stabilizes, it and we have $Z^m(A, b) \sim Z^m(A + C, b + d)$.

2. STABILIZATION AND REGULARIZATION

The matrices $C(A, \lambda, \tau)$ and $d(b, \lambda, \tau)$ considered in Sec. 1 allow one to use the $(C(A, \lambda, \tau), d(b, \lambda, \tau))$-operator in what follows.

THEOREM 1. The $(C(A, \lambda, \tau), d(b, \lambda, \tau))$-operator stabilizes a vector nontrivial IQP problem $Z^m(A, b)$, $m \geq 2$, for any $\lambda \in \mathbf{R}^m_>$ and $\tau > 0$ and, at the same time, we have

$$\mathrm{Sl}^m(A + C(A, \lambda, \tau), b + d(b, \lambda, \tau)) \subseteq P^m(A, b). \tag{5}$$

Proof. Taking into account the nontriviality of the problem $Z^m(A, b)$, we conclude that there is a vector $x \in \overline{P^m}(A, b)$. Then we have $\pi(x, A, b) \neq \varnothing$ and, for any solution $x' \in \pi(x, A, b)$, using inequalities (4), we obtain

$$q_i(x, x', A_i + C_i, b_i + d_i) = q_i(x, x', A_i, b_i) + q_i(x, x', C_i, d_i)$$

$$\geq q_i(x, x', C_i, d_i) = \tau \sum_{k=1}^{m} \lambda_k q_k(x, x', A_k, b_k) > 0, \ i \in N_m.$$

Therefore, according to the definition of the set $\sigma(x, A + C(A, \lambda, \tau), b + d(b, \lambda, \tau))$, we have $x' \in \sigma(x, A + C(A, \lambda, \tau), b + d(b, \lambda, \tau))$. Thus, the following relationships are true:

$$\forall x \in \overline{P^m}(A, b) \ \ \pi(x, A, b) \subseteq \sigma(x, A + C(A, \lambda, \tau), b + d(b, \lambda, \tau)).$$

From this, by virtue of Lemma 1, we derive inclusion (5).

Next, according to Lemma 3, there is a number $\varepsilon > 0$ such that we have

$$\forall (A', b') \in \Xi(\varepsilon) \ \ (\mathrm{Sl}^m(A + C(A, \lambda, \tau) + A', b + d(b, \lambda, \tau) + b')$$

$$\subseteq \mathrm{Sl}^m(A + C(A, \lambda, \tau), b + d(b, \lambda, \tau))).$$

Hence, taking into account inclusions (1) and (5), we conclude that

$$\exists \varepsilon > 0 \ \ \forall (A', b') \in \Xi(\varepsilon) \ \ (P^m(A + C(A, \lambda, \tau) + A', b + d(b, \lambda, \tau) + b') \subseteq P^m(A, b)),$$

i.e., the $(C(A, \lambda, \tau), d(b, \lambda, \tau))$-operator stabilizes the problem $Z^m(A, b)$.

Theorem 1 is proved.

Hereafter, we will use the denotations

$$\Delta_{\min} = \min\{q_i(x, x', A_i, b_i) > 0 : x, x' \in X, i \in N_m\}, \quad \Delta_{\max} = \max\{\| x - x' \|_*^2 : x, x' \in X\},$$

$$\gamma = \frac{\Delta_{\min}}{m \|\lambda\| (\|A\| + \|b\|) \Delta_{\max}}.$$

Since the problem $Z^m(A, b)$ is assumed to be nontrivial, we have $\|A\| + \|b\| > 0$ and $\Delta_{\min} > 0$.

THEOREM 2. For any vector $\lambda \in \mathbf{R}_{>}^{m}$ and any number τ that satisfies the inequalities

$$0 < \tau < \gamma, \tag{6}$$

the $(C(A,\lambda,\tau), d(b,\lambda,\tau))$-operator regularizes a vector nontrivial IQP problem $Z^m(A,b)$, $m \geq 2$.

Proof. According to Theorem 1, the $(C(A,\lambda,\tau), d(b,\lambda,\tau))$-operator stabilizes the problem $Z^m(A,b)$. Therefore, to prove Theorem 2, it remains to show that we have

$$Z^m(A + C(A,\lambda,\tau), b + d(b,\lambda,\tau)) \sim Z^m(A,b). \tag{7}$$

Inclusions (1) and (5) imply the inclusion

$$P^m(A + C(A,\lambda,\tau), b + d(b,\lambda,\tau)) \subseteq P^m(A,b). \tag{8}$$

We prove the opposite inclusion using the proof from the contrary. We assume that there is a point $x \in P^m(A,b) \setminus P^m(A + C(A,\lambda,\tau), b + d(b,\lambda,\tau))$. Then the set $\pi(x, A + C(A,\lambda,\tau), b + d(b,\lambda,\tau))$ is nonempty. If x' is an element of this set, then the inequality $q(x,x', A + C(A,\lambda,\tau), b + d(b,\lambda,\tau)) \neq \mathbf{0}$ is true. Therefore, according to Lemma 4, $q(x,x', A,b) \neq \mathbf{0}$, i.e., since we have $x \in P^m(A,b)$, an index $s \in N_m$ can be found such that we obtain $q_s(x,x', A_s, b_s) < 0$. From this, successively applying formula (4), Lemma 2, and inequalities (6), we derive

$$q_s(x,x', A_s + C_s, b_s + d_s) = q_s(x,x', A_s, b_s) + q_s(x,x', C_s, d_s)$$

$$= q_s(x,x', A_s, b_s) + \tau \sum_{k=1}^{m} \lambda_k q_k(x,x', A_k, b_k) \leq q_s(x,x', A_s, b_s) + \tau \sum_{k=1}^{m} |\lambda_k q_k(x,x', A_k, b_k)|$$

$$\leq q_s(x,x', A_s, b_s) + \tau m \|\lambda\| (\|A\| + \|b\|) \|x - x'\|_*^2 < q_s(x,x', A_s, b_s) + \gamma m \|\lambda\| (\|A\| + \|b\|) \Delta_{\max}$$

$$= q_s(x,x', A_s, b_s) + \Delta_{\min} \leq q_s(x,x', A_s, b_s) + |q_s(x,x', A_s, b_s)| = 0.$$

This means that we obtain $x' \notin \pi(x, A + C(A,\lambda,\tau), b + d(b,\lambda,\tau))$. The obtained contradiction together with inclusion (8) proves equivalence (7).

Theorem 2 is proved.

3. ε-STABILIZATION AND ε-REGULARIZATION

Let $\varepsilon > 0$, let $A, C \in \mathbf{R}^{m \times n \times n}$, b, and let $d \in \mathbf{R}^{m \times n}$. We say that the (C,d)-operator ε-stabilizes a problem $Z^m(A,b)$ if formula (3) is fulfilled.

In addition to the denotations $\Delta_{\min}$, $\Delta_{\max}$, and γ introduced above, we will also use

$$\delta_{\min} = \min \left\{ \sum_{k=1}^{m} \lambda_k q_k(x,x', A_k, b_k) > 0 : x, x' \in X \right\}.$$

Since the problem $Z^m(A,b)$ is assumed to be nontrivial by assumption, we have $\delta_{\min} > 0$ for $\lambda \in \mathbf{R}_{>}^{m}$.

THEOREM 3. For any $\varepsilon > 0$, $\lambda \in \mathbf{R}_{>}^{m}$, and also

$$\tau \geq \frac{\varepsilon \Delta_{\max}}{\delta_{\min}}, \tag{9}$$

the $(C(A,\lambda,\tau), d(b,\lambda,\tau))$-operator ε-stabilizes a vector nontrivial IQP problem $Z^m(\overline{A,b})$, $m \geq 2$.

Proof. In view of relations (4) and (9) and Lemma 2, for any $(A', b') \in \Xi(\varepsilon)$, $x \in P^m(A,b)$, and $x' \in \pi(x, A, b)$, the following inequalities are fulfilled:

$$q_i(x, x', A_i + C_i + A'_i, b_i + d_i + b'_i) = q_i(x, x', A_i, b_i) + q_i(x, x', C_i, d_i) + q_i(x, x', A'_i, b'_i)$$

$$\geq q_i(x, x', C_i, d_i) + q_i(x, x', A'_i, b'_i) = \tau \sum_{k=1}^{m} \lambda_k q_k(x, x', A_k, b_k) + q_i(x, x', A'_i, b'_i)$$

$$\geq \tau\delta_{\min} - (\|A'\| + \|b'\|)\|x - x'\|_*^2 > \tau\delta_{\min} - \varepsilon\Delta_{\max} \geq 0, \ i \in N_m.$$

Therefore, we obtain $\forall (A', b') \in \Xi(\varepsilon) \ (\mathrm{Sl}^m(A + C(A, \lambda, \tau) + A', b + d(b, \lambda, \tau) + b') \subseteq P^m(A, b))$.

To complete the proof, it remains to use inclusion (1).

Theorem 3 is proved.

We call a problem $Z^m(A, b)$ ε-stable if formula (2) is fulfilled.

Let $\varepsilon > 0$. We say that the (C, d)-operator ε-regularizes a problem $Z^m(A, b)$ if we have $Z^m(A, b) \sim Z^m(A + C, b + d)$ and the problem $Z^m(A + C, b + d)$ is ε-stable.

THEOREM 4. For any vector $\lambda \in \mathbf{R}_>^m$ and any numbers ε and τ that satisfy the inequalities

$$0 < \varepsilon < \frac{\gamma\delta_{\min}}{\Delta_{\max}}, \tag{10}$$

$$\frac{\varepsilon\Delta_{\max}}{\delta_{\min}} \leq \tau < \gamma, \tag{11}$$

the $(C(A, \lambda, \tau), d(b, \lambda, \tau))$-operator ε-regularizes a vector nontrivial IQP problem $Z^m(A, b)$, $m \geq 2$.

Proof. We first note the consistency of conditions (10) and (11) in the sense that, for any number ε specified by inequalities (10), there is a number τ that satisfies condition (11).

Since any number τ specified by inequalities (11) also satisfies inequality (9), according to Theorem 3, for any number ε specified by conditions (10), the $(C(A, \lambda, \tau), d(b, \lambda, \tau))$-operator ε-stabilizes the problem $Z^m(A, b)$ for τ from conditions (11). This means that we have

$$\forall (A', b') \in \Xi(\varepsilon) \ (P^m(A + C(A, \lambda, \tau) + A', b + d(b, \lambda, \tau) + b') \subseteq P^m(A, b)). \tag{12}$$

Since inequality (11) also implies inequalities (6), Theorem 2 implies the truth of the equality $P^m(A, b) = P^m(A + C(A, \lambda, \tau), b + d(b, \lambda, \tau))$.

Taking into account this fact and formula (12), we conclude that the $(C(A, \lambda, \tau), d(b, \lambda, \tau))$-operator ε-regularizes the vector IQP problem $Z^m(A, b)$.

Theorem 4 is proved.

Comment. In view of the controllability of the vector parameter λ used in defining γ, the ε-regularization of the vector IQP problem $Z^m(A, b)$ can be performed for any number $\varepsilon > 0$. It is obvious that the increase in the number ε leads to increasing the stability radius of the problem $Z^m(A + C(A, \lambda, \tau), b + d(b, \lambda, \tau))$ (the definition of the stability radius of a problem see in [1, 2, 11, 13]).

REFERENCES

1. I. V. Sergienko, L. N. Kozeratskaya, and T. T. Lebedeva, Investigation and Parametric Analysis of Discrete Optimization Problems [in Russian], Naukova Dumka, Kiev (1995).
2. I. V. Sergienko and V. P. Shilo, Discrete Optimization Problems: Issues, Solution Methods, and Investigations [in Russian], Naukova Dumka, Kiev (2003).
3. E. G. Belousov and V. G. Andronov, Solvability and Stability of Problems of Polynomial Programming [in Russian], Izd. MGU, Moscow (1993).
4. T. T. Lebedeva and T. I. Sergienko, "Comparative analysis of different types of stability with respect to constraints of a vector integer-optimization problem," Cybernetics and Systems Analysis, No. 1, 52–57 (2004).

5. T. T. Lebedeva, N. V. Semenova, and T. I. Sergienko, "Stability of vector problems of integer optimization: Relationship with the stability of sets of optimal and nonoptimal solutions," Cybernetics and Systems Analysis, No. 4, 551–558 (2005).
6. T. T. Lebedeva and T. I. Sergienko, "Stability of a vector integer quadratic programming problem with respect to vector criterion and constraints," Cybernetics and Systems Analysis, No. 5, 667–674 (2006).
7. S. A. Ashmanov, Linear Programming [in Russian], Nauka, Moscow (1981).
8. Yu. A. Dubov, S. I. Travkin, and V. N. Yakimets, Multicriteria Models for Formation and Choice of Variants of Systems [in Russian], Nauka, Moscow (1986).
9. L. N. Kozeratskaya, T. T. Lebedeva, and T. I. Sergienko, "Regularization of integer vector optimization problems," Cybernetics and Systems Analysis, No. 3, 455–458 (1993).
10. V. A. Emelichev and O. A. Yanushkevich, "Regularization of a multicriterial integer linear programming problem," Izv. Vuzov, Matematika, No. 12, 38–42 (1999).
11. V. A. Emelichev, E. Girlich, Yu. V. Nikulin, and D. P. Podkopaev, "Stability and regularization of vector problems of integer linear programming," Optimization, **51**, No. 4, 645–676 (2002).
12. V. A. Emelichev and O. A. Yanushkevich, "Regularization of a lexicographic vector problem of integer programming," Cybernetics and Systems Analysis, No. 6, 951–955 (1999).
13. V. A. Emelichev and O. A. Yanushkevich, "Stability and regularization of the lexicographic vector problem of quadratic discrete programming," Cybernetics and Systems Analysis, No. 2, 196–202 (2000).
14. N. Z. Shor and S. I. Stetsenko, Quadratic Extremal Problems and Nondifferentiable Optimization [in Russian], Naukova Dumka, Kiev (1989).