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BRIEF COMMUNICATIONS

STABILITY OF A COMBINATORIAL VECTOR PARTITION PROBLEM

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The vector variant of the partition problem is considered. It is shown that the coincidence of the Pareto and Slater sets is the necessary and sufficient condition of stability of the problem with respect to its functional.

Keywords: *multicriteriality, partition problem, stability.*

As is well known [1] (see also [2–4]), the coincidence of Pareto and Slater sets is the necessary and sufficient condition of stability (with respect to the functional) of a vector (multicriteria) problem of integer linear programming with a finite set of feasible solutions. In this paper, we prove that the coincidence of these sets is also the criterion of stability of the vector variant of the classical partition problem that is as follows: divide a set of several numbers into two subsets so that the sums of elements in the subsets are minimal. In the case when the elements of a set are positive, this problem is equivalent to the scheduling-theory problem that consists of the distribution of independent tasks between two identical processors so that the moment of time at which the last fulfilled task comes to an end is minimal [5]. In scheduling theory, this problem is denoted by $P|\cdot|C_{\max}$.

By an m-criterion (vector) partition problem

$$Z^m(C): \min\{f(x,C): x \in Q^n\},$$

where $f(x,C) = (|C_1 x|, |C_2 x|, \dots, |C_m x|)$, we understand the problem of searching for the set of efficient solutions (the Pareto set)

$$P^m(C) = \{x \in Q^n : \pi(x,C) = \varnothing\}.$$

Here, $\pi(x,C) = \{x' \in X : f(x,C) \geq f(x',C) \,\&\, f(x,C) \neq f(x',C)\}$, C_i is the ith row of a matrix $C = [c_{ij}]_{m\times n} \in \mathbf{R}^{m\times n}$, $m \geq 1$, $n \geq 2$, $i \in N_m = \{1, 2, \dots, m\}$, $x = (x_1, x_2, \dots, x_n)^T \in Q^n$, and $Q = \{-1, 1\}$.

By virtue of the finiteness of Q^n, the set $P^m(C)$ is nonempty for any matrix $C \in \mathbf{R}^{m\times n}$. It is obvious that, when $x^0 \in P^m(C)$, the solution $-x^0$ is also efficient in view of the equality $f(x^0, C) = f(-x^0, C)$. Therefore, the Pareto set of the problem $Z^m(C)$ always consists of an even number of solutions.

Let us investigate the stability of the Pareto set $P^m(C)$ with respect to perturbations of the parameters of the vector function $f(x,C)$ by addition of perturbation matrices to the matrix C. To this end, we specify a metric l_∞ in a space $\mathbf{R}^k$ of arbitrary dimension $k \in \mathbf{N}$, i.e., the norm of a vector $z = (z_1, z_2, \dots, z_k) \in \mathbf{R}^k$ is understood to be the following number:

$$||z||_\infty = \max_{j \in N_k} |z_j|$$

and the norm of a matrix is understood to be the norm of the vector composed of its elements. We also use the

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following denotation for any $t \in \mathbf{R}$:

$$\operatorname{sg} t = \begin{cases} 1 & \text{if } t \geq 0, \\ -1 & \text{if } t < 0. \end{cases}$$

For any number $\varepsilon > 0$, we introduce the following set of perturbation matrices in the space $\mathbf{R}^{m \times n}$ with the metric l_∞:

$$\Omega(\varepsilon) = \{C' \in \mathbf{R}^{m \times n} : ||C'||_\infty < \varepsilon\}.$$

Following [1–4, 6], we call the problem $Z^m(C)$ stable (with respect to the functional) if we have

$$\Xi(C) = \{\varepsilon > 0 : \forall\, C' \in \Omega(\varepsilon)\, (P^m(C + C') \subseteq P^m(C))\} \neq \varnothing.$$

Hence, the stability of the problem is a discrete analogue of the property of Hausdorff upper semicontinuity at a point $C \in \mathbf{R}^{m \times n}$ of an optimal multivalued mapping $P^m : \mathbf{R}^{m \times n} \to 2^{Q^n}$, i.e., a point-set mapping that associates the Pareto set $P^m(C)$ with each set of parameters (the matrix C) of the problem.

It is well known that if the equality $P^m(C) = Q^n$ is true, then the problem $Z^m(C)$ is stable. In what follows, we exclude this case from consideration and call the problem $Z^m(C)$ for which the set $\overline{P}^m(C) := Q^n \setminus P^m(C)$ is nonempty nontrivial.

We introduce the set $\mathrm{Sl}^m(C)$ of weakly-efficient solutions (the Slater set) by the rule

$$x \in \mathrm{Sl}^m(C) \Leftrightarrow \forall\, x' \in Q^n \quad \exists\, i \in N_m \quad (|C_i x| \leq |C_i x'|). \tag{1}$$

It is obvious that the inclusion $P^m(C) \subseteq \mathrm{Sl}^m(C)$ is true for any matrix $C \in \mathbf{R}^{m \times n}$. Moreover, if we have $x \in \mathrm{Sl}^m(C)$, then we also have $-x \in \mathrm{Sl}^m(C)$.

For any index $i \in N_m$, we put $X_i = \{x \in Q^n : C_i x \neq 0\}$.

As is obvious, for a nontrivial problem $Z^m(C)$, the set $M_m = \{i \in N_m : X_i \neq \varnothing\}$ is nonempty. Therefore, for such problems, we have

$$\Delta(C) = \min_{i \in M_m} \min_{x \in X_i} |C_i x| > 0. \tag{2}$$

LEMMA. Let x^* be a weakly-efficient solution of a nontrivial problem $Z^m(C)$. Then, for any number $\varepsilon > 0$, there is a matrix $C^* \in \Omega(\varepsilon)$ such that we have $x^* \in P^m(C + C^*)$.

Proof. Let $\varepsilon > 0$. We construct the elements of the perturbation matrix $C^* = [c_{ij}^*] \in \mathbf{R}^{m \times n}$ by the rule

$$c_{ij}^* = \sigma_i^* x_j^* \gamma_{ij}, \quad (i, j) \in N_m \times N_n, \tag{3}$$

where $\sigma_i^* = \operatorname{sg} C_i x^*$ and

$$\gamma_{ij} = \begin{cases} \left(1 - \dfrac{1}{n}\right)\delta & \text{if } C_i x^* = 0,\ j = n, \\ -\dfrac{\delta}{n} & \text{in the other cases,} \end{cases} \qquad 0 < \delta < \min\{\varepsilon, \Delta(C)\}. \tag{4}$$

Then it is obvious that we have $C^* \in \Omega(\varepsilon)$. If we have $x \in Q^n \setminus \{x^*, -x^*\}$, then, in view of rule (1), an index $p = p(x) \in N_m$ can be found such that we obtain

$$|C_p x^*| \leq |C_p x|. \tag{5}$$

Let us consider two possible cases.

Case 1. $C_p^* x \neq 0$. Then, by virtue of (2)–(4), we found

$$\delta < \Delta(C) \leq |C_p x^*|, \quad C_p^* x^* = -\sigma_p^* \delta, \quad |C_p^* x| < \delta. \tag{6}$$

Therefore, we have

$$|(C_p + C_p^*)x^*| = |C_p x^* - \sigma_p^* \delta| = |C_p x^*| - \delta > 0, \ |(C_p + C_p^*)x| \geq |C_p x| - |C_p^* x|.$$

From this, using inequalities (5) and (6), we obtain

$$|(C_p + C_p^*)x^*| - |(C_p + C_p^*)x| \leq |C_p x^*| - \delta - |C_p x| + |C_p^* x| < 0.$$

Case 2. $C_p x^* = 0$. Then, taking into account the structure of the vector C_p^* specified by formulas (3) and performing simple computations, we make sure that the equality $C_p^* x^* = 0$ is true and, based on this equality, we have

$$(C_p + C_p^*)x^* = 0. \tag{7}$$

It is also easy to verify that the following inequalities are true

$$0 < |C_p^* x| \leq \delta < \Delta(C). \tag{8}$$

Next, we will consider two possibilities.

1. $C_p x = 0$. Here, using equality (7) and inequalities (8), we obtain

$$|(C_p + C_p^*)x^*| - |(C_p + C_p^*)x| = -|C_p^* x| < 0.$$

2. $C_p x \neq 0$. In this case, inequality (2) implies that $\Delta(C) \leq |C_p x|$.

From this, according to equality (7) and inequalities (8), we obtain

$$|(C_p + C_p^*)x^*| - |(C_p + C_p^*)x| = -|(C_p + C_p^*)x| \leq |C_p^* x| - |C_p x| \leq \delta - \Delta(C) < 0.$$

Summarizing the aforesaid, we conclude that we have

$$\forall \varepsilon > 0 \ \ \exists C^* \in \Omega(\varepsilon), \ \ \forall x \in Q^n \setminus \{x^*, -x^*\} \ \ \exists p \in N_m \ \ \left(|(C_p + C_p^*)x^*| < |(C_p + C_p^*)x| \right).$$

In view of the obvious truth of the equality $|(C_p + C_p^*)x^*| = |(C_p + C_p^*)(-x^*)|$, the inclusion $x^* \in P^m(C + C^*)$ is true, which implies the truth of the lemma.

THEOREM. A nontrivial problem $Z^m(C)$, $m \geq 1$, is stable if and only if we have

$$P^m(C) = \mathrm{Sl}^m(C). \tag{9}$$

Proof. Sufficiency. Let equality (9) be true. Then, in view of the nontrivial character of the problem $Z^m(C)$, we have

$$\forall x \in \overline{P}^m(C) \ \ \exists \widetilde{x} \in Q^n \quad \forall i \in N_m \quad (|C_i x| > |C_i \widetilde{x}|).$$

Therefore, by virtue of the continuity of the function $f(x, C)$ on the set of matrices $C \in \mathbf{R}^{m \times n}$, a number $\varepsilon = \varepsilon(x) > 0$ exists such that, for any matrix $C^* \in \Omega(\varepsilon)$, the following inequalities hold true:

$$|(C_i + C_i^*)x| > |(C_i + C_i^*)\widetilde{x}|, \quad i \in N_m.$$

This implies the inclusion $x \in \overline{P}^m(C + C^*)$.

Hence, for any solution $x \in \overline{P}^m(C)$, a number

$$\varepsilon^* = \min\{\varepsilon(x) : x \in \overline{P}^m(C)\},$$

can be found such that, for any perturbation matrix $C^* \in \Omega(\varepsilon^*)$, the inequality $\pi(x, C + C^*) \neq \varnothing$ is true, which means that we have $x \in \overline{P}^m(C + C^*)$, i.e., the inclusion $P^m(C + C^*) \subseteq P^m(C)$ is true. Therefore, the problem $Z^m(C)$ is stable.

We prove necessity by contradiction. Let $P^m(C) \neq \mathrm{Sl}^m(C)$ under the assumption that the problem $Z^m(C)$ is stable. Then a solution $x^* \in \mathrm{Sl}^m(C) \setminus P^m(C)$ can be found. Therefore, by virtue of the lemma, for any number $\varepsilon > 0$, there exists a matrix $C^* \in \Omega(\varepsilon)$ such that we have $x^* \in P^m(C + C^*)$.

Thus, the formula

$$\exists\, x^* \in \overline{P}^m(C) \quad \forall\, \varepsilon > 0 \quad \exists\, C^* \in \Omega(\varepsilon) \quad (x^* \in P^m(C + C^*))$$

that testifies to the truth of $P^m(C + C^*) \not\subseteq P^m(C)$ is true. Hence, the problem $Z^m(C)$ is not stable.

The theorem is proved.

When we pass to the one-criterion case ($m = 1$), the set of efficient solutions is transformed into the set of optimal solutions $P^1(C)$ of a scalar partition problem $Z^1(C)$, $C \in \mathbf{R}^n$. Since we have $P^1(C) = \mathrm{Sl}^1(C)$, the corollary formulated below follows from the theorem.

COROLLARY. The scalar problem $Z^1(C)$ is stable for any vector $C \in \mathbf{R}^n$.

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