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On stability and quasi-stability radii for a vector combinatorial problem with a parametric optimality principle

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Abstract. A vector combinatorial linear problem with a parametric optimality principle that allows us to relate the well-known choice functions of jointly-extremal and Pareto solution is considered. A quantitative analysis of stability for the set of generalized efficient trajectories under the independent perturbations of coefficients of linear functions is performed. Formulas of stability and quasi-stability radii are obtained in the l_∞-metric. Some results published earlier are derived as corollaries.

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1 Problem statement

Let us consider a typical vector (n-criteria) combinatorial problem. Assume that, on the system of subsets (trajectories) $T \subseteq 2^{N_m}$, $|T| \geq 2$, $N_m = \{1, 2, \dots, m\}$, $m \geq 2$, a vector criterion

$$f(t, A) = (f_1(t, A), f_2(t, A), \dots, f_n(t, A)) \to \min_{t \in T}$$

is defined. Here

$$f_i(t, A) = \sum_{j \in t} a_{ij}, \ i \in N_n, \ n \geq 1$$

are the linear partial criteria, where $A = [a_{ij}]_{n \times m} \in \mathbf{R}^{n \times m}$, $n, m \in \mathbf{N}$. Assume that $f_i(\emptyset, A) = 0$.

Now we introduce the binary relation $\succ$, in the space $\mathbf{R}^d$ of any dimension $d \in \mathbf{N}$, which generates the Pareto optimality principle [1], assuming that, for any different vectors $y = (y_1, y_2, \dots, y_d)$ and $y' = (y'_1, y'_2, \dots, y'_d)$ of the space the formula

$$y \succ y' \ \Leftrightarrow \ y \geq y' \ \& \ y \neq y'$$

holds.

Let $s \in N_n$, $N_n = \bigcup_{r \in N_s} J_r$ be the partitioning of the set N_n into s nonempty nonintersecting groups, i. e. $J_r \neq \emptyset$, $r \in N_s$; $p \neq q \Rightarrow J_p \cap J_q = \emptyset$. For this partitioning define the set $T^n(A, J_1, J_2, \ldots, J_s)$ of *generalized efficient*, or, in other words of $(J_1, J_2, \ldots, J_s)$-*efficient* trajectories according to the formula

$$T^n(A, J_1, J_2, \ldots, J_s) = \{t \in T: \ \exists k \in N_s \ \ \forall t' \in T \ \ (f_{J_k}(t, A) \overline{\succ} f_{J_k}(t', A))\},$$

where $\overline{\succ}$ denotes the negation of relation $\succ$, $f_{J_k}(t, A)$ is the projection of the vector $f(t, A)$ onto the coordinate axes of the space $\mathbf{R}^n$ with the numbers of group J_k.

It is evident that N_n-efficient trajectory $t \in T^n(A, N_n)$ $(s = 1)$ is a Pareto optimal trajectory on the set of trajectories T. Therefore, it is easy to see that the set of N_n-efficient trajectories $T^n(A, N_n)$ is Pareto set

$$P^n(A) = \{t \in T: \ \forall t' \in T \ \ (f(t, A) \overline{\succ} f(t', A))\}.$$

Clearly, in another extreme case, where $s = n$, the set of trajectories $T^n(A, \{1\}, \{2\}, \ldots, \{n\})$ is the set of *jointly-extremal* trajectories

$$C^n(A) = \{t \in T: \ \exists k \in N_n \ \ \forall t' \in T \ \ (f_k(t, A) \leq f_k(t', A))\}$$

(see, for example, [2, 3]).

In this context, by the parametrization of the principle of optimality we mean introducing a characteristic of binary relation of preference that allows us to relate the well-known choice functions of jointly-extremal and Pareto solution.

Denote the vector problem of finding $T^n(A, J_1, J_2, \ldots, J_s)$ by $Z^n(A, J_1, J_2, \ldots, J_s)$. It is evident that $T^1(A, N_1)$ is the set of optimal trajectories of the scalar (single criterion) linear combinatorial problem $Z^1(A, N_1)$, where $A \in \mathbf{R}^m$, in scheme of which many extremal graph, boolean programming and scheduling theory problems are put in.

2 Stability radius

Following [4–10], the *stability radius* of $Z^n(A, J_1, J_2, \ldots, J_s)$ is the number

$$\rho_1^n(A, J_1, J_2, \ldots, J_s) = \begin{cases} \sup \Xi_1 & \text{if } \ \Xi_1 \neq \emptyset, \\ 0 & \text{in other cases,} \end{cases}$$

where $\Xi_1 = \{\varepsilon > 0: \ \forall B \in \Omega(\varepsilon) \ \ (T^n(A+B, J_1, J_2, \ldots, J_s) \subseteq T^n(A, J_1, J_2, \ldots, J_s))\}$, $\Omega(\varepsilon) = \{B \in \mathbf{R}^{n \times m}: \ \ ||B|| < \varepsilon\}$, $||B|| = \max\{|b_{ij}|: \ (i, j) \in N_n \times N_m\}$, $B = [b_{ij}]_{n \times m}$.

In other words, the stability radius of $Z^n(A, J_1, J_2, \ldots, J_s)$ determines the limiting level of perturbations of elements of A of payoff function in the l_∞-metric, for which new $(J_1, J_2, \ldots, J_s)$-efficient trajectories do not appear. Obviously, $Z^n(A, J_1, J_2, \ldots, J_s)$ is stable and the stability radius is infinite if the equality $T^n(A, J_1, J_2, \ldots, J_s) = T$ holds. If the set

$$\overline{T^n}(A, J_1, J_2, \ldots, J_s) := T \setminus T^n(C, J_1, J_2, \ldots, J_s)$$

is nonempty, then we say that $Z^n(A, J_1, J_2, \dots, J_s)$ is *non-trivial.*

For any nonempty set $J \subseteq N_n$ we introduce the notation

$$P(A, J) = \{t \in T : \ \forall t' \in T \ \ (f_J(t, A) \overline{\succ} f_J(t', A))\}.$$

Then we have

$$P(A, N_n) = P^n(A),$$

$$T^n(A, J_1, J_2, \dots, J_s) = \{t \in T : \ \exists k \in N_s \ \ (t \in P(A, J_k))\}. \tag{1}$$

Suppose

$$\Delta(t, t') = |(t \cup t') \setminus (t \cap t')|,$$

$$g_i(t, t', A) = f_i(t, A) - f_i(t', A).$$

Henceforth we will use the following evident inequality

$$g_i(t, t', A) \le ||A||\Delta(t, t'). \tag{2}$$

Theorem 1. *For the stability radius $\rho_1^n(A, J_1, J_2, \dots, J_s)$ of the nontrivial problem $Z^n(A, J_1, J_2, \dots, J_s)$, $n \ge 1$, $s \ge 1$, the following formula is valid*

$$\rho_1^n(A, J_1, J_2, \dots, J_s) = \min_{k \in N_s} \ \min_{t \in \overline{T^n}(A, J_1, J_2, \dots, J_s)} \ \max_{t' \in T^n(A, J_1, J_2, \dots, J_s)} \ \min_{i \in J_k} \frac{g_i(t, t', A)}{\Delta(t, t')}. \tag{3}$$

Proof. Note that due to the nontriviality of $Z^n(A, J_1, J_2, \dots, J_s)$ the set $\overline{T^n}(A, J_1, J_2, \dots, J_s)$ is nonempty.

Let us introduce the notations: ρ_1 and φ are accordingly the left-hand and the right-hand sides of equality (3).

It is easy to see that $\varphi \ge 0$. At first we prove the inequality $\rho_1 \ge \varphi$. If $\varphi = 0$, then this inequality is obvious. Let $\varphi > 0$, $B \in \Omega(\varphi)$, $t \in \overline{T^n}(A, J_1, J_2, \dots, J_s)$. Let us show that $t \in \overline{T^n}(A + B, J_1, J_2, \dots, J_s)$.

It follows directly from the definition of φ that for any $k \in N_s$ and $t \in \overline{T^n}(A, J_1, J_2, \dots, J_s)$ there exists trajectory $t^* \in T^n(A, J_1, J_2, \dots, J_s)$ such that for any indices $i \in J_k$ the inequality $g_i(t, t^*, A) \ge \varphi\Delta(t, t^*)$ holds.

Hence, taking into account (2), we derive

$$g_i(t, t^*, A + B) = g_i(t, t^*, A) + g_i(t, t^*, B) \ge \varphi\Delta(t, t^*) - ||B||\Delta(t, t^*) > 0, \ i \in J_k.$$

Therefore we have $f_{J_k}(t, A + B) \succ f_{J_k}(t^*, A + B)$, $k \in N_s$, i. e. $t \in \overline{T^n}(A + B, J_1, J_2, \dots, J_s)$.

Thus, the formula

$$\forall B \in \Omega(\varphi) \ \ (T^n(A + B, J_1, J_2, \dots, J_s) \subseteq T^n(A, J_1, J_2, \dots, J_s))$$

holds, and as consequence, $\rho_1 \ge \varphi$.

Now let us show that $\rho_1 \leq \varphi$. According to the definition of φ there exist $k \in N_s$ and $t^0 \in \overline{T^n}(A, J_1, J_2, \dots, J_s)$ such that for any trajectory $t' \in T^n(A, J_1, J_2, \dots, J_s)$ there exists an index $p = p(t') \in J_k$ such that

$$g_p(t^0, t', A) \leq \varphi\Delta(t^0, t').$$

Then, assuming $\varepsilon > \varphi$, $B^0 = [b^0_{ij}]_{n\times m} \in \Omega(\varepsilon)$, where

$$b^0_{ij} = \begin{cases} \alpha & \text{if } \ i \in J_k,\ j \notin t^0, \\ -\alpha & \text{if } \ i \in J_k,\ j \in t^0, \\ 0 & \text{in other cases,} \end{cases}$$

$$\varphi < \alpha < \varepsilon,$$

and using (2), we derive

$$g_p(t^0, t', A + B^0) = g_p(t^0, t', A) + g_p(t^0, t', B^0) \leq \varphi\Delta(t^0, t') - \alpha\Delta(t^0, t') < 0,$$

i. e. we have

$$\forall\varepsilon > \varphi \ \ \exists B^0 \in \Omega(\varepsilon) \ \ \forall t' \in T^n(A, J_1, J_2, \dots, J_s) \ \ (f_{J_k}(t^0, A + B^0) \overline{\succ} f_{J_k}(t', A + B^0)). \tag{4}$$

Consider two possible cases.

Case 1. $t^0 \in T^n(A + B^0, J_1, J_2, \dots, J_s)$. Then, using the inclusion $t^0 \in \overline{T^n}(A, J_1, J_2, \dots, J_s)$, we derive

$$\forall\varepsilon > \varphi \ \ \exists B^0 \in \Omega(\varepsilon) \ \ (T^n(A + B^0, J_1, J_2, \dots, J_s) \not\subseteq T^n(A, J_1, J_2, \dots, J_s)). \tag{5}$$

Case 2. $t^0 \in \overline{T^n}(A + B^0, J_1, J_2, \dots, J_s)$. Then $t^0 \notin P(A + B^0, J_k)$ and due to the external stability [11] of Pareto set $P(A + B^0, J_k)$ there exists a trajectory $t^* \in P(A + B^0, J_k)$, such that $f_{J_k}(t^0, A + B^0) \succ f_{J_k}(t^*, A + B^0)$. Hence, according to (4) we have $t^* \in \overline{T^n}(A, J_1, J_2, \dots, J_s)$ and taking into account (1) we obtain $t^* \in T^n(A + B^0, J_1, J_2, \dots, J_s)$. Therefore formula (5) holds.

Summarizing these two cases, we conclude that for any $\varepsilon > \varphi$ we have $\rho_1 < \varepsilon$. Consequently, $\rho_1 \leq \varphi$. □

Theorem 1 implies the following results known earlier.

Corollary 1 [5]. *For the stability radius of the nontrivial problem $Z^n(A, N_n)$ with Pareto optimality principle the following formula*

$$\rho_1^n(A, N_n) = \min_{t \in \overline{P^n}(A)} \ \max_{t' \in P^n(A)} \ \min_{i \in N_n} \frac{g_i(t, t', A)}{\Delta(t, t')} \tag{6}$$

holds, where $\overline{P^n}(A) = T \setminus P^n(A)$.

Corollary 2 [12]. *For the stability radius of the nontrivial problem* $Z^n(A, \{1\}, \{2\}, \ldots, \{n\})$ *with jointly-extremal optimality principle the following formula*

$$\rho_1^n(A, \{1\}, \{2\}, \ldots, \{n\}) = \min_{i \in N_n} \min_{t \in \overline{C^n}(A)} \max_{t' \in C^n(A)} \frac{g_i(t, t', A)}{\Delta(t, t')} \tag{7}$$

holds, where $\overline{C^n}(A) = T \setminus C^n(A)$.

The partial case of the formulas (6) and (7) is the well-known formula of the stability radius of the scalar ($n = 1$) linear trajectory problem [4,6].

3 Quasi-stability radius

As usual (see [5, 7, 13, 14]), the *quasi-stability radius* of $Z^n(A, J_1, J_2, \ldots, J_s)$ is defined as

$$\rho_2^n(A, J_1, J_2, \ldots, J_s) = \begin{cases} \sup \Xi_2 & \text{if } \ \Xi_2 \neq \emptyset, \\ 0 & \text{in other cases,} \end{cases}$$

where

$$\Xi_2 = \{\varepsilon > 0 : \ \forall B \in \Omega(\varepsilon) \ \ (T^n(A, J_1, J_2, \ldots, J_s) \subseteq T^n(A + B, J_1, J_2, \ldots, J_s))\}.$$

Thus, the quasi-stability radius of $Z^n(A, J_1, J_2, \ldots, J_s)$ is the limit level of independent perturbations of elements of A, for which the generalized efficient trajectories of initial problem do not disappear.

Theorem 2. *For the quasi-stability radius* $\rho_2^n(A, J_1, J_2, \ldots, J_s)$ *of the problem* $Z^n(A, J_1, J_2, \ldots, J_s)$*,* $n \geq 1$*,* $s \geq 1$*, the following formula is valid*

$$\rho_2^n(A, J_1, J_2, \ldots, J_s) = \min_{t' \in T^n(A, J_1, J_2, \ldots, J_s)} \max_{k \in N_s} \min_{t \in T \setminus \{t'\}} \max_{i \in J_k} \frac{g_i(t, t', A)}{\Delta(t, t')}. \tag{8}$$

Proof. Let us introduce the notations: ρ_2 and ξ are accordingly the left-hand and the right-hand sides of equality (8).

It is easy to see that $\xi \geq 0$. At first we prove the inequality $\rho_2 \geq \xi$. If $\xi = 0$, then this inequality is obvious. Let $\xi > 0$, $B \in \Omega(\xi)$.

It follows from the definition of ξ that for any trajectory $t' \in T^n(A, J_1, J_2, \ldots, J_s)$ there exists $k \in N_s$ such that for any trajectory $t \in T \setminus \{t'\}$ there exists $p = p(t) \in J_k$ such that

$$g_p(t, t', A) \geq \xi \Delta(t, t').$$

Hence, taking into account (2), we derive

$$g_p(t, t', A + B) = g_p(t, t', A) + g_p(t, t', B) \geq \xi \Delta(t, t') - ||B|| \Delta(t, t') > 0.$$

Therefore, we have $f_{J_k}(t', A + B) \overline{\succ} f_{J_k}(t, A + B)$. Thus, we have proved the formula

$$\forall B \in \Omega(\xi) \ \forall t' \in T^n(A, J_1, J_2, \ldots, J_s) \ \exists k \in N_s \ \forall t \in T \ \ (f_{J_k}(t', A{+}B) \overline{\succ} f_{J_k}(t, A{+}B)),$$

which implies

$$\forall B \in \Omega(\xi) \ \ (T^n(A, J_1, J_2, \ldots, J_s) \subseteq T^n(A + B, J_1, J_2, \ldots, J_s)),$$

and therefore the inequality $\rho_2 \geq \xi$ holds.

Now we show that $\rho_2 \leq \xi$. According to the definition of ξ there exists a trajectory $t^0 \in T^n(A, J_1, J_2, \ldots, J_s)$ such that for any $k \in N_s$ there exists a trajectory $t^* \in T \setminus \{t^0\}$ such that

$$\forall i \in J_k \ \ (g_i(t^*, t^0, A) \leq \xi\Delta(t^*, t^0)).$$

Then, assuming $\varepsilon > \xi$, $\widehat{B} = [\widehat{b}_{ij}]_{n\times m} \in \Omega(\varepsilon)$, where

$$\widehat{b}_{ij} = \begin{cases} \alpha & \text{if } \ i \in N_n, \ j \in t^0, \\ -\alpha & \text{if } \ i \in N_n, \ j \notin t^0, \end{cases}$$

$$\xi < \alpha < \varepsilon,$$

and taking into account (2), we derive

$$g_i(t^*, t^0, A + \widehat{B}) = g_i(t^*, t^0, A) + g_i(t^*, t^0, \widehat{B}) \leq \xi\Delta(t^*, t^0) - \alpha\Delta(t^*, t^0) < 0, \ i \in J_k,$$

i. e. $f_{J_k}(t^0, A + \widehat{B}) \succ f_{J_k}(t^*, A + \widehat{B})$. Thus, we have proved the following formula

$$\forall \varepsilon > \xi \ \ \exists \widehat{B} \in \Omega(\varepsilon) \ \ \forall k \in N_s \ \ \exists t^* \in T \ \ (f_{J_k}(t^0, A + \widehat{B}) \succ f_{J_k}(t^*, A + \widehat{B})),$$

which implies

$$T^n(A, J_1, J_2, \ldots, J_s) \not\subseteq T^n(A + \widehat{B}, J_1, J_2, \ldots, J_s).$$

It follows that the quasi-stability radius ρ_2 does not exceed ξ. □

Corollary 3 [13]. *For the quasi-stability radius of the problem $Z^n(A, N_n)$ with Pareto optimality principle the following formula is valid*

$$\rho_2^n(A, N_n) = \min_{t' \in P^n(A)} \ \min_{t \in T\setminus\{t'\}} \ \max_{i \in N_n} \ \frac{g_i(t, t', A)}{\Delta(t, t')}.$$

Corollary 4. *For the quasi-stability radius of the problem $Z^n(A, \{1\}, \{2\}, \ldots, \{n\})$ with jointly-extremal optimality principle the following formula is valid*

$$\rho_2^n(A, \{1\}, \{2\}, \ldots, \{n\}) = \min_{t' \in C^n(A)} \ \max_{i \in N_n} \ \min_{t \in T\setminus\{t'\}} \ \frac{g_i(t, t', A)}{\Delta(t, t')}.$$

In conclusion we note that the analogous quantitative characteristics of different stability types of discrete and game theory problems with another kinds of parametrization of optimality principles were considered in the works [8–10, 14–16].

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