Ann Oper Res (2012) 201:265–286
DOI 10.1007/s10479-012-1203-5

Balancing of simple assembly lines under variations of task processing times

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Published online: 7 September 2012

Abstract One of the simple assembly line balancing problems (SALBPs), known as SALBP-E, is considered. It consists in assigning a given set $V = \{1, 2, \ldots, n\}$ of elementary tasks to linearly ordered workstations with respect to precedence and capacity restrictions while minimizing the following product: number of used workstations $\times$ working time on the most loaded one. The stability of feasible and optimal solutions for this problem with regard to possible variations of the processing time of certain tasks is investigated. Two heuristic procedures finding a compromise between the efficiency and the considered stability measure of studied solutions are suggested and evaluated on known benchmarks.

Keywords Stability/sensitivity analysis · Uncertainty · Assembly line balancing problems

1 Introduction

A simple assembly line is a typical flow-oriented production system that consists of a number of workstations ($m \geq 2$) aligned in a serial manner along a conveyer belt without buffers between them. All workstations function simultaneously performing elementary tasks assigned to them. Tasks can be executed by an operator or using special automatic machines installed at workstations. Identical product items are consequently launched down the line and processed at every workstation in the order of its location. A workstation operates only one product item at a time. In the common case of paced assembly lines, all product items situated on the line are moved to the next respective workstation at the same time by some kind of transportation system.

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All elementary tasks required for completing a product item constitute a given set $V = \{1, 2, \dots, n\}$ associated with a vector $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n_+$ of task processing times, where t_j is the processing time of task $j \in V$ and $\mathbb{R}_+$ is the set of all positive real numbers. One principal design aim for such production systems is to assign set V of tasks to workstations with respect to certain constraints while optimizing one or several objectives.

The set of tasks assigned to workstation k is denoted by V_k ($V_k \subset V$) determining its load. The sum of task processing times from set V_k ($t(V_k) = \sum_{j \in V_k} t_j$) defines the *working time* on workstation k and must not be greater than *line cycle time* c, i.e. $t(V_k) \leq c$. It is evident that $c \geq t_{\max} = \max\{t_j : j \in V\}$. A workstation with the greatest working time is called *most loaded*.

According to an assembly process, tasks cannot be carried out in an arbitrary sequence, but are subject to *precedence constraints*. This type of constraints can be represented by an oriented acyclic graph $G = (V, \mathcal{A})$, where arc (i, j) belongs to $\mathcal{A}$ if task i cannot be executed after task j.

One of the important issues of managing assembly lines is the balancing problem. With regard to objectives used, simple assembly line balancing problems (SALBPs) are commonly classified (Rekiek et al. 2002; Tasan and Tunali 2008) into three types: minimize the total number of opened workstations for a fixed line cycle time c (SALBP-1); minimize the working time on the most loaded workstation with a fixed number of workstations m (SALBP-2); and if neither number of workstations nor line cycle time is fixed, maximize the *line efficiency* (SALBP-E). The latter objective consists in minimizing the following product: number of opened workstations × working time on the most loaded one. It should be emphasized that all these problems are known to be $\mathcal{NP}$-hard (Scholl 1999, Chap. 2.2.1.5).

Note that SALBP-1 and SALBP-2 have been more intensively investigated than SALBP-E (studied in this paper). One of the sparse but quite natural search methods for SALBP-E was proposed in Rosenblatt and Carlson (1985), where an exact method for solving SALBP-1 with $c = t_{\max}$ is firstly applied. As a result, the minimal number m of workstations is obtained for that problem. This number can be obviously considered as an upper bound on the number of workstations for an optimal solution of SALBP-E. Thus, the first product $m \cdot t_{\max}$ is determined. After that sequentially decreasing m by one and applying an exact method for solving the corresponding SALBP-2, $m - 2$ respective products are obtained. The minimal product value among $m - 1$ ones found in this way corresponds to an optimal solution.

Currently, one of the important topics under assembly line design and balancing is the task processing time variability engendered by the following frequent factors: instability of operators performing tasks with respect to work rate, skill, and motivation; different material composition of product items; changes in product and workstation characteristics; as well as failure sensitivity of complex processes (workstation micro-stopping and breakdowns, tasks execution inaccuracy). The works studying the processing time variability and dealing with different types of ALBPs usually consider the following models: *stochastic processing times* (Ağpak and Gökçen 2007; Baykasoğlu and Özbakır 2007; Chiang and Urban 2006; Erel et al. 2005; Gamberini et al. 2009; Liu et al. 2005; Urban and Chiang 2006); *fuzzy processing times* (Gen et al. 1996; Hop 2006; Tsujimura et al. 1995). However, applying these models in practice relates to some difficulties in determining an appropriate probability or possibility distribution function for each task processing time.

In contrast to stochastic and fuzzy models, another approach was proposed in Sotskov et al. (2006) for SALBP-1, where the influence of variations of task processing times (VTPT) for optimal solutions already found for an original deterministic problem was investigated. The main advantage in applying of this *post-optimal* approach compared with the mentioned above models is that it is sufficient to know only the set of tasks whose processing time can vary. However, it can be successfully applied if the level of VTPT is not very high. The principal goal of this approach is to determine the limit level of independent VTPT (named the *stability radius*) for an optimal solution such that it remains optimal. The stability radius is an appropriate measure of the credibility of found solutions in presence of VTPT. Thus, for example, there is no need to reconstruct already found optimal solution if the VTPT not greater than its stability radius. Otherwise it can lose its optimality and even feasibility, and the construction of approximate solutions may become a more reasonable decision.

Notice that a similar approach was already studied for different types of combinatorial optimization problems such as lot-sizing problem (Van Hoesel and Wagelmans 1993), traveling salesman problem (Libura et al. 1998), minimum spanning tree problem (Pettie 2005), knapsack problem (Belgacem and Hifi 2008), scheduling problems (Guinand et al. 2004; Hall and Posner 2004; Petrovic et al. 2008; Sotskov et al. 1998, 2010); as well as for the general form of integer and Boolean linear optimization problems (Emelichev et al. 2002; Emelichev and Podkopaev 2010; Kılınç-Karzan et al. 2009; Libura 1999; Libura and Nikulin 2006).

In this paper, the investigation of stability for both feasible and optimal solutions for SALBP-E is presented. The rest of the paper is organized as follows. In Sect. 2 basic definitions and properties are introduced. A short illustrative example is presented in Sect. 3. Sections 4, 5 and 6 are devoted to the calculation of the stability radius for feasible, quasi-feasible (see the definition in Sect. 2), and optimal solutions, respectively. Heuristic procedures finding a compromise between the efficiency and the value of the stability radius of a feasible solution is described in Sect. 7. Experimental results carried out on known benchmarks constitute in Sect. 8. Final remarks and conclusions are given in Sect. 9.

2 Basic definitions and properties

For the basic version of SALBP-E, the following parameters are usually given: $c \in \mathbb{R}_+$ and $m \in \mathbb{N}$ are maximal authorized values on the workstation working time and on the number of workstations, respectively, that impose the so-called *capacity constraints*.

Definition 1 An assignment of tasks V to workstations is called a feasible balance if neither precedence nor capacity constraints are violated.

Hereafter, the set of feasible balances for a given vector $t \in \mathbb{R}^n_+$ is denoted as $\mathbf{B}_{\mathcal{F}}(t)$, where each balance b is characterized by set $\{V^b_1, V^b_2, \dots, V^b_{m^b}\}$ of non-intersecting nonempty subsets of V such that $V = V^b_1 \cup V^b_2 \cup \dots \cup V^b_{m^b}$ determining set $W^b = \{1, 2, \dots, m^b\}$ of workstations in b.

The goal of SALBP-E can be expressed as follows:

$$\mathcal{Z}(b,t) := m^b \cdot c(b,t) \to \min_{b \in \mathbf{B}_{\mathcal{F}}(t)},$$

where $c(b,t) = \max\{t(V^b_k) : k \in W^b\}$ is the working time on the most loaded workstation of balance b for vector t. Feasible balances with the minimum value of goal function $\mathcal{Z}(b,t)$ are called optimal balances which constitute set $\mathbf{B}_{\mathcal{O}}(t)$. Obviously $\mathbf{B}_{\mathcal{O}}(t) \subseteq \mathbf{B}_{\mathcal{F}}(t)$.

Two following evident properties will be used in the further presentation.

Property 1 *For any two balances* $b, b^0 \in \mathbf{B}_{\mathcal{O}}(t)$: $\mathcal{Z}(b,t) = \mathcal{Z}(b^0,t)$.

Property 2 *For $b \in \mathbf{B}_{\mathcal{O}}(t)$ and $b^0 \in \mathbf{B}_{\mathcal{F}}(t) \setminus \mathbf{B}_{\mathcal{O}}(t)$: $\mathcal{Z}(b,t) < \mathcal{Z}(b^0,t)$.*

In this paper, we consider that set V contains two types of tasks:

- Uncertain tasks: their processing time can vary during the line life cycle. The set of such tasks is denoted by $\widetilde{V}$.
- Precise tasks: their processing time remains the same during the line life cycle. Such tasks constitute set $V \setminus \widetilde{V}$.

Without loss of generality, suppose that $\widetilde{V} = \{1, 2, \dots, \tilde{n}\}$ and $V \setminus \widetilde{V} = \{\tilde{n}+1, \tilde{n}+2, \dots, n\}$, where $0 < \tilde{n} \le n$.

VTPT from set $\widetilde{V}$ can be represented by vector $\xi = (\xi_1, \xi_2, \dots, \xi_{\tilde{n}}, 0, 0, \dots, 0) \in \mathbb{R}^n$, where ξ_j, $j \in \widetilde{V}$, can be both positive or negative. Thus, the vector of task processing times in a certain moment of the line life can be represented by the *perturbed* vector $t^* = (t_1 + \xi_1, t_2 + \xi_2, \dots, t_{\tilde{n}} + \xi_{\tilde{n}}, t_{\tilde{n}+1}, \dots, t_n)$.

Remark 1 In this study, it is supposed that $t_j^* = \max\{0, t_j + \xi_j\}$, $j \in V$.

Note that VTPT does not modify either the precedence constraints or the number of workstations of an optimal or feasible balance, however, they can affect its optimality and even feasibility. The feasibility of a balance can be lost, if the working time on the most loaded workstation becomes greater than c for a new perturbed vector t^*. An optimal balance b found for original vector t may lose its optimality for some new perturbed vector t^*, if there is a balance $b^0 \in \mathbf{B}_{\mathcal{F}}(t^*)$ such that $\mathcal{Z}(b^0, t^*) < \mathcal{Z}(b, t^*)$.

Note also that balance b^0 respecting the precedence constraints and having $m^{b^0} \le m$, but $c(b^0, t) > c$ can become feasible and even optimal for a new perturbed vector t^*, if $c(b^0, t^*) \le c$. The set of such balances is denoted by $\mathbf{B}_{\widehat{\mathcal{F}}}(t)$ and its elements are called *quasi-feasible* balances.

To correctly model possible VTPT, the Chebyshev distance between two vectors t and t' from $\mathbb{R}^n_+$ is introduced:

$$\|t - t'\| = \max\{|t_i - t_i'| : i \in V\}.$$

This induces the notion of *ε-neighborhood* of t over $\mathbb{R}^n_+$:

$$\Omega(\varepsilon, t) = \{t' \in \Psi(t) : \|t - t'\| < \varepsilon\}, \quad \varepsilon > 0,$$

where

$$\Psi(t) = \{t' \in \mathbb{R}^n_+ : t_j' = t_j, j \in V \setminus \widetilde{V}\}.$$

Introducing $\mathcal{R} \in \{\mathcal{F}, \widehat{\mathcal{F}}, \mathcal{O}\}$, the following definitions are in the center of this study, where $\mathcal{F}$, $\widehat{\mathcal{F}}$ and $\mathcal{O}$ designate the feasibility, quasi-feasibility and optimality, respectively.

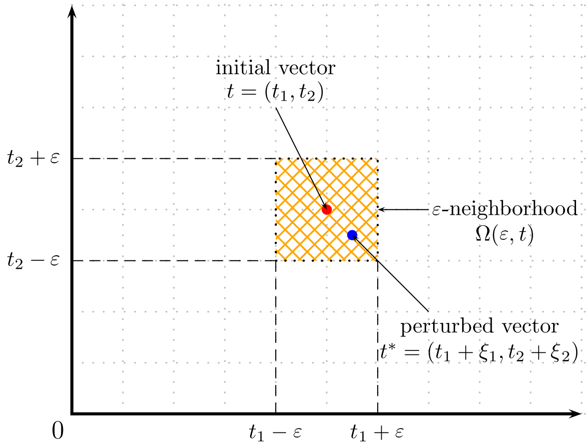
Definition 2 Balance $b \in \mathbf{B}_{\mathcal{R}}(t)$ is called $\mathcal{R}$-*stable* if there exists an ε-neighborhood $\Omega(\varepsilon, t)$ such that for any $t' \in \Omega(\varepsilon, t)$, b remains in $\mathbf{B}_{\mathcal{R}}(t')$, i.e. b is $\mathcal{R}$-stable if the following condition holds:

$$\Xi_{\mathcal{R}}(b, t) := \{\varepsilon > 0 : \forall t' \in \Omega(\varepsilon, t)(b \in \mathbf{B}_{\mathcal{R}}(t'))\} \ne \emptyset.$$

Definition 3 $\mathcal{R}$-*stability radius* $\rho_{\mathcal{R}}(b, t)$ of balance $b \in \mathbf{B}_{\mathcal{R}}(t)$ is defined as the least upper limit of $\Xi_{\mathcal{R}}(b, t)$, i.e.

$$\rho_{\mathcal{R}}(b, t) = \sup \Xi_{\mathcal{R}}(b, t).$$

Fig. 1 Perturbation modeling



Remark 2 In this study, it is supposed that $\sup \emptyset = 0$.

A two-dimensional example of the introduced notions is given in Fig. 1.

The $\mathcal{R}$-stability radius of balance $b \in \mathbf{B}_{\mathcal{R}}(t)$ can be considered as the maximal radius of an opened ball over $(\mathbb{R}^n_+, \|\cdot\|)$ with the center at point t such that b remains in $\mathbf{B}_{\mathcal{R}}(t')$ whatever a perturbed vector t' within this ball.

It is easy to see that b is $\mathcal{R}$-stable (not $\mathcal{R}$-stable) iff $\rho_{\mathcal{R}}(b,t) > 0$ ($\rho_{\mathcal{R}}(b,t) = 0$); and $\rho_{\mathcal{O}}(b,t) \le \rho_{\mathcal{F}}(b,t)$ holds for any optimal balance b.

The goal of this paper is to evaluate the complexity of the $\mathcal{R}$-stability radius calculation and investigate the trade-off between its value and the efficiency of a balance studied.

In further, the following evident properties are used.

Property 3 *If inequality $\mathcal{Z}(b,t) < \mathcal{Z}(b^0,t)$ holds for balances b and b^0, then the following is true*:

$$\exists \varepsilon > 0\ \forall t' \in \Omega(\varepsilon, t) \quad \bigl(\mathcal{Z}\bigl(b, t'\bigr) < \mathcal{Z}\bigl(b^0, t'\bigr)\bigr).$$

Property 4 *For any balance b the following is true*:

$$\forall \varepsilon > 0\ \forall t' \in \Omega(\varepsilon, t)\ \forall k \in \widetilde{W}^b \quad \bigl(t\bigl(V_k^b\bigr) - \varepsilon\bigl|\widetilde{V}_k^b\bigr| < t'\bigl(V_k^b\bigr) < t\bigl(V_k^b\bigr) + \varepsilon\bigl|\widetilde{V}_k^b\bigr|\bigr).$$

Here $\widetilde{W}^b = \{k \in W^b : \widetilde{V}_k^b \neq \emptyset\}$, $\widetilde{V}_k^b = V_k^b \cap \widetilde{V}$, $t'(V_k^b) = \sum_{j \in V_k^b} t'_j$.

Suppose

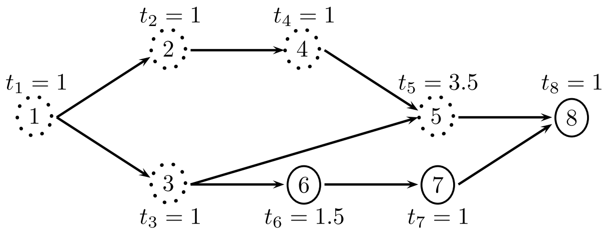
$$K(b,t) = \bigl\{k \in W^b : t\bigl(V_k^b\bigr) = c(b,t)\bigr\}. \tag{1}$$

In other words, $K(b,t)$ is the set of the most loaded workstations of balance b for vector t. Obviously $K(b,t)$ is a non-empty set. For any feasible balance b such that $W^b \neq K(b,t)$, the following parameter is introduced:

$$\Delta(b) = \min\left\{\frac{t(V_k^b) - t(V_l^b)}{|\widetilde{V}_k^b| + |\widetilde{V}_l^b|} : k \in K(b,t), l \in W^b \setminus K(b,t)\right\}. \tag{2}$$

Note that for any $k \in K(b,t)$ and $l \in W^b \setminus K(b,t)$, $t(V_k^b) - t(V_l^b) > 0$. Therefore, since $\widetilde{W}^b \neq \emptyset$, then there is at least one pair of indices $k^* \in K(b,t)$ and $l^* \in W^b \setminus K(b,t)$ such

Fig. 2 Precedence constraints



that $|\widetilde{V}_{k^*}^b| + |\widetilde{V}_{l^*}^b| \neq 0$ and, as a consequence, $\Delta(b) > 0$. It is supposed that $\Delta(b) = +\infty$ in the case $W^b = K(b,t)$.

Property 5 *For any feasible balance b the following formula holds*:

$$\forall t' \in \Omega\big(\Delta(b), t\big) \quad \Big(c\big(b, t'\big) = \max_{k \in K(b,t)} t'\big(V_k^b\big)\Big). \tag{3}$$

Proof Note that if $W^b = K(b,t)$ formula (3) is obvious. Therefore, hereafter we suppose that $W^b \neq K(b,t)$. Using the definition of $\Delta(b)$, we have

$$\forall k \in K(b,t)\ \forall l \in W^b \setminus K(b,t) \quad \big(\Delta(b)\big(\big|\widetilde{V}_k^b\big| + \big|\widetilde{V}_l^b\big|\big) \leq t\big(V_k^b\big) - t\big(V_l^b\big)\big). \tag{4}$$

Whence, using Property 4, we derive

$$\min_{k \in K(b,t)} t'\big(V_k^b\big) \underbrace{\geq}_{\text{Property } 4} \min_{k \in K(b,t)} \big\{t\big(V_k^b\big) - \Delta(b)\big|\widetilde{V}_k^b\big|\big\}$$
$$\underbrace{\geq}_{(4)} \max_{l \in W^b \setminus K(b,t)} \big\{t\big(V_l^b\big) + \Delta(b)\big|\widetilde{V}_l^b\big|\big\} \underbrace{\geq}_{\text{Property } 4} \max_{l \in W^b \setminus K(b,t)} t'\big(V_l^b\big),$$

where at least the first or the third inequality is strict that implies (3). □

This property shows that if $k \notin K(b,t)$, then $t'(V_k^b) < c(b,t')$ for any perturbed vector t' within $\Delta(b)$-neighborhood. Therefore, as this will be shown below, the stability of a studied balance closely depends on the assignment of tasks to workstations from $K(b,t)$.

3 Example

In this section, an example is used to illustrate the notations introduced just before.

Let $V = \{1, 2, \ldots, 8\}$, $\widetilde{V} = \{1, 2, \ldots, 5\}$, $t = (1, 1, 1, 1, 3.5, 1.5, 1, 1)$, $c = 5$, $m = 5$. The precedence constraints are represented by the acyclic direct graph shown in Fig. 2, where the tasks from $\widetilde{V}$ are dotted.

Let b^1 be a feasible balance such that $V_1^{b^1} = \{1, 2, 3, 4\}$, $V_2^{b^1} = \{5\}$, $V_3^{b^1} = \{6, 7, 8\}$. It is easy to calculate that $t(V_1^{b^1}) = 4$, $t(V_2^{b^1}) = 3.5$, $t(V_3^{b^1}) = 3.5$. Therefore, $c(b,t) = \max\{t(V_1^{b^1}), t(V_2^{b^1}), t(V_3^{b^1})\} = 4$, i.e. $\mathcal{Z}(b^1, t) = 12$.

Let b^2 be another feasible balance such that $V_1^{b^2} = \{1, 3, 6\}$, $V_2^{b^2} = \{2, 4, 7\}$, $V_3^{b^2} = \{5, 8\}$. As a consequence, $t(V_1^{b^2}) = 3.5$, $t(V_2^{b^2}) = 3$, $t(V_3^{b^2}) = 4.5$. Similarly, we obtain $c(b,t) = \max\{t(V_1^{b^2}), t(V_2^{b^2}), t(V_3^{b^2})\} = 4.5$, $\mathcal{Z}(b^2, t) = 13.5$.

Using (1), $K(b^1, t) = \{1\}$ and $K(b^2, t) = \{3\}$, and according to (2):

$$\Delta(b^1) = \min\left\{\frac{4-3.5}{4+1}, \frac{4-3.5}{4+0}\right\} = \frac{1}{10},$$
$$\Delta(b^2) = \min\left\{\frac{4.5-3.5}{1+2}, \frac{4.5-3}{1+2}\right\} = \frac{1}{3}.$$

This means that within $\frac{1}{10}$-neighborhood and $\frac{1}{3}$-neighborhood of t the first and the third workstations remain the most loaded ones for b^1 and b^2, respectively.

Notice also that though $\mathcal{Z}(b^1,t) < \mathcal{Z}(b^2,t)$ the second balance is more interesting with respect to the $\mathcal{F}$-stability radius, since $\rho_{\mathcal{F}}(b^1,t) = 0.25 < \rho_{\mathcal{F}}(b^2,t) = 0.5$ (see Theorem 1). Below Sect. 7 establishes a compromise between these two antagonistic objectives.

In the next section the behavior of feasible balances under VTPT is evaluated.

4 Feasible balances

Recall that balance $b \in \mathbf{B}_{\mathcal{F}}(t)$ can lose its feasibility for a new perturbed vector $t^* \in \mathbb{R}^n_+$, i.e. $b \notin \mathbf{B}_{\mathcal{F}}(t^*)$, only if the working time on its most loaded workstation becomes greater than c, i.e. if $c(b,t^*) > c$.

The following theorem presents the formula of the $\mathcal{F}$-stability radius calculation determined as the limit level of VTPT such that a studied feasible balance preserves its feasibility within the bounds of this level.

Theorem 1 *$\mathcal{F}$-stability radius $\rho_{\mathcal{F}}(b,t)$ of balance $b \in \mathbf{B}_{\mathcal{F}}(t)$ is calculated as follows*:

$$\rho_{\mathcal{F}}(b,t) = \min_{k\in\widetilde{W}^b} \frac{c - t(V_k^b)}{|\widetilde{V}_k^b|}. \tag{5}$$

Proof To simplify further the statement, the following notation is introduced: ρ and φ are the left-hand and the right-hand sides of (5), respectively. Obviously, φ is a nonnegative finite number due to inclusion $b \in \mathbf{B}_{\mathcal{F}}(t)$.

To prove formula (5), we consequently show that inequalities $\rho \geq \varphi$ and $\rho \leq \varphi$ hold.

First let us prove that $\rho \geq \varphi$. To do this, it is sufficient to check that

$$\forall t' \in \Omega(\varphi,t) \quad \big(b \in \mathbf{B}_{\mathcal{F}}(t')\big). \tag{6}$$

If $\varphi = 0$ inequality $\rho \geq \varphi$ is evident. Let $\varphi > 0$, $t' \in \Omega(\varphi,t)$. By definition of φ we have

$$\forall k \in \widetilde{W}^b \quad \big(\varphi\big|\widetilde{V}_k^b\big| \leq c - t\big(V_k^b\big)\big).$$

Whence, using Property 4, we obtain

$$c - t'\big(V_k^b\big) > c - \big(t\big(V_k^b\big) + \varphi\big|\widetilde{V}_k^b\big|\big) \geq 0, \quad k \in \widetilde{W}^b.$$

Therefore, taking into account the following obvious inequalities $c \geq t'(V_k^b)$, $k \in W^b \setminus \widetilde{W}^b$, we conclude that

$$\forall t' \in \Omega(\varphi,t)\ \forall k \in W^b \quad \big(c \geq t'\big(V_k^b\big)\big).$$

In other words, $c \geq c(b,t')$, $t' \in \Omega(\varphi,t)$, i.e. formula (6) holds.

Now let us show that $\rho \leq \varphi$. The proof of the latter inequality is equivalent to the proof of the following formula:

$$\forall \varepsilon > \varphi\ \exists t^* \in \Omega(\varepsilon,t) \quad \big(b \notin \mathbf{B}_{\mathcal{F}}\big(t^*\big)\big). \tag{7}$$

To prove formula (7), the definition of φ is used. By definition of φ, there is $k^* \in \widetilde{W}^b$ such that $\varphi|\widetilde{V}^b_{k^*}| = c - t(V^b_{k^*})$. Then, assuming $\varepsilon > \varphi$, $t^* \in \Omega(\varepsilon, t)$, where

$$t^*_j = \begin{cases} t_j + \delta, & \text{if } j \in \widetilde{V}^b_{k^*}, \\ t_j & \text{otherwise}, \end{cases}$$
$$\varphi < \delta < \varepsilon,$$

we obtain

$$c - t^*\big(V^b_{k^*}\big) = c - \big(t\big(V^b_{k^*}\big) + \delta\big|\widetilde{V}^b_{k^*}\big|\big) = (\varphi - \delta)\big|\widetilde{V}^b_{k^*}\big| < 0.$$

It follows that $c(b, t^*) > c$, i.e. $b \notin \mathbf{B}_{\mathcal{F}}(t^*)$, and therefore (7) holds. □

Theorem 1 implies

Corollary 1 *Balance $b \in \mathbf{B}_{\mathcal{F}}(t)$ is not $\mathcal{F}$-stable iff the following is true*:

$$\exists k \in \widetilde{W}^b \quad \big(c = t\big(V^b_k\big)\big).$$

Proof *Sufficiency*. Let $k \in \widetilde{W}^b$ such that $c = t(V^b_k)$. Then from (5) we have $\rho_{\mathcal{F}}(b, t) = 0$ and therefore b is not $\mathcal{F}$-stable.

Necessity. Let $b \in \mathbf{B}_{\mathcal{F}}(t)$ be not $\mathcal{F}$-stable, then $\rho_{\mathcal{F}}(b, t) = 0$. Whence, using (5), we conclude that there is $k \in \widetilde{W}^b$ such that $c = t(V^b_k)$. □

Corollary 2 *The problem of finding $\mathcal{F}$-stability radius of balance $b \in \mathbf{B}_{\mathcal{F}}(t)$ is polynomially solvable*.

Proof An algorithm of finding $\rho_{\mathcal{F}}(b, t)$ consists in the sequential analysis of the workstations of balance b and can be described as follows: for current workstation $k \in W^b$ calculate expression $\varphi(k) = \frac{c - t(V^b_k)}{|\widetilde{V}^b_k|}$, if $\widetilde{V}^b_k \neq \emptyset$ (this takes $\mathcal{O}(|V^b_k|)$ computing time), otherwise the next workstation is analyzed. This continues until either all workstations are analyzed or workstation $k \in \widetilde{W}^b$ such that $\varphi(k) = 0$ is found. In the latter case, $\rho_{\mathcal{F}}(b, t) = 0$, otherwise $\rho_{\mathcal{F}}(b, t) = \min_{k \in \widetilde{W}^b} \varphi(k)$. Thus, in the worst case, where $\widetilde{W}^b = W^b$ and no $\varphi(k)$, $k \in W^b$, equals 0, this algorithm takes linear time: $\mathcal{O}(\sum_{k=1}^{m^b} |V^b_k|) = \mathcal{O}(|V|) = \mathcal{O}(n)$. □

Note that the introduced definition of stability is close to the definition of robustness (see, e.g., Billaut et al. 2008) which can be stated as follows.

A studied solution of an optimization problem is called robust if small variations of the initial data of this problem imply small changes of the solution performance.

Following this general definition, the $\mathcal{F}$-stability radius calculated in polynomial time can be considered as an appropriate measure of robustness of feasible balances. Since greater the $\mathcal{F}$-stability radius becomes, more robust a feasible balance is under VTPT.

The next section addresses to evaluate the $\widehat{\mathcal{F}}$-stability radius for quasi-feasible balances.

5 Quasi-feasible balances

Recall that for any balance $b \in \mathbf{B}_{\widehat{\mathcal{F}}}(t)$, inequality $c(b, t) > c$ holds. Despite of this, a quasi-feasible balance may become a feasible one. This may occur if for a new perturbed vector $t^* \in \mathbb{R}^n_+$, $c(b, t^*) \leq c$. However, if for a quasi-feasible balance there is a workstation without uncertain tasks and whose working time exceeds c, then this balance always remains quasi-feasible.

In the sequel, the set of workstations of balance b with the working time exceeding c is denoted by $\widehat{W}^b$.

Theorem 2 *For the $\widehat{\mathcal{F}}$-stability radius $\rho_{\widehat{\mathcal{F}}}(b,t)$ of balance $b \in \mathbf{B}_{\widehat{\mathcal{F}}}(t)$, the following inequality holds*:

$$\rho_{\widehat{\mathcal{F}}}(b,t) \geq \max_{k \in \widehat{W}^b} \frac{t(V_k^b) - c}{|\widetilde{V}_k^b|}. \tag{8}$$

Proof As in Theorem 1, the supplementary notation is introduced: ρ and φ are the left-hand and the right-hand sides of (8), respectively. It is easy to see that φ can be equal to $+\infty$. This may occur if $\widetilde{V}_k^b = \emptyset$ for some index $k \in \widehat{W}^b$ and means that balance b will never become feasible. Otherwise φ is a positive finite number due to inclusion $b \in \mathbf{B}_{\widehat{\mathcal{F}}}(t)$. Therefore, hereafter we suppose that $\widetilde{V}_k^b \neq \emptyset$ for any $k \in \widehat{W}^b$.

To prove inequality (8), we show that inequality $\rho \geq \varphi$ holds. To do this, it is sufficient to check that

$$\forall t' \in \Omega(\varphi,t) \quad \big(b \in \mathbf{B}_{\widehat{\mathcal{F}}}\big(t'\big)\big). \tag{9}$$

If $\varphi = 0$ inequality $\rho \geq \varphi$ is evident. Let $\varphi > 0$, $t' \in \Omega(\varphi,t)$. By definition of φ we have

$$\exists k^* \in \widehat{W}^b \quad \big(\varphi\big|\widetilde{V}_{k^*}^b\big| = t\big(V_{k^*}^b\big) - c\big).$$

Whence, using Property 4, we obtain

$$t'\big(V_{k^*}^b\big) - c > \big(t\big(V_{k^*}^b\big) - \varphi\big|\widetilde{V}_{k^*}^b\big|\big) - c = 0.$$

It follows that $c(b,t') > c$, $t' \in \Omega(\varphi,t)$, i.e. formula (9) holds. □

Theorem 2 implies

Corollary 3 *Any quasi-feasible balance is $\widehat{\mathcal{F}}$-stable.*

Proof Since for any balance $b \in \mathbf{B}_{\widehat{\mathcal{F}}}(t)$ the following inequalities $c > t(V_k^b)$, $k \in \widehat{W}^b$, hold, then we have $\rho_{\widehat{\mathcal{F}}}(b,t) > 0$ due to inequality (8). □

Corollary 3 means that there is $\varepsilon^* > 0$ such that, whatever a perturbed vector t' within $\Omega(\varepsilon^*,t)$, no quasi-feasible balance becomes feasible one. As a consequence, the $\widehat{\mathcal{F}}$-stability radius of a quasi-feasible balance is never equal to zero.

6 Optimal balances

The present section addresses to the stability of optimal balances under VTPT. The goal is to find the conditions of existence of an ε-neighborhood of t (i.e. $\Omega(\varepsilon,t)$) such that, whatever a perturbed vector t' within this neighborhood, an optimal balance remains optimal, i.e. belongs to $\mathbf{B}_{\mathcal{O}}(t')$.

In fact, there are two principal cases, where an optimal balance b can lose its optimality for a new perturbed vector t^*:

- b loses its feasibility (this case was studied in Sect. 4);
- b remains feasible, but there is $b^0 \in \mathbf{B}_{\mathcal{F}}(t^*)$ such that $\mathcal{Z}(b^0,t^*) < \mathcal{Z}(b,t^*)$.

Following Properties 2, 3 and Corollary 3, the second case may occur under small VTPT only if b^0 is also an optimal balance of the initial problem, i.e. $b^0 \in \mathbf{B}_{\mathcal{O}}(t)$.

Let us introduce the following notation

$$\Upsilon(b,t) = \{\widetilde{V}_k^b : k \in K(b,t)\},$$

where $\Upsilon(b,t)$ represents the set that consists of sets of uncertain tasks assigned to the most loaded workstations. Obviously $\Upsilon(b,t) \neq \emptyset$, but can contain the empty set as its element.

Using evident inequality $m^b \cdot c(b,t) \geq t(V)$, $b \in \mathbf{B}_{\mathcal{F}}(t)$, we have $m_{\min} \geq \lceil \frac{t(V)}{c} \rceil$, where $m_{\min} = \min\{m^b : b \in \mathbf{B}_{\mathcal{F}}(t)\}$.

Remark 3 To simplify the description of the obtained results, it is supposed that $m < 2m_{\min}$.

Lemma 1 *If for* $b, b^0 \in \mathbf{B}_{\mathcal{O}}(t)$ *inclusion* $\Upsilon(b,t) \subseteq \Upsilon(b^0,t)$ *does not hold, then* b *is not* $\mathcal{O}$*-stable.*

Proof If inclusion $\Upsilon(b,t) \subseteq \Upsilon(b^0,t)$ does not hold, then there exists at least one $f \in K(b,t)$ such that set $\widetilde{V}_f^b$ does not belong to set $\Upsilon(b^0,t)$.

Two cases are possible.

Case 1: $\widetilde{V}_f^b = \emptyset$. This means that for any $k \in K(b^0,t)$ we have $\widetilde{V}_k^{b^0} \neq \emptyset$.

Then, taking

$$t_j^* = \begin{cases} t_j - \delta, & \text{if } j \in \widetilde{V}, \\ t_j & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon < \min\{\Delta(b), \Delta(b^0)\}$, we obtain that $t^* \in \Omega(\varepsilon,t)$.

According to Properties 1 and 5, we get

$$\begin{aligned} m^b \cdot c(b,t^*) &\underbrace{=}_{\text{Property 5}} m^b \cdot \max_{k\in K(b,t)} \sum_{j\in V_k^b} t_j^* = m^b \cdot \max\Big\{\sum_{j\in V_f^b} t_j^*, \max_{k\in K(b,t)\setminus\{f\}} \sum_{j\in V_k^b} t_j^*\Big\} \\ &\underbrace{=}_{\widetilde{V}_f^b=\emptyset} m^b \cdot \max\Big\{\sum_{j\in V_f^b} t_j, \max_{k\in K(b,t)\setminus\{f\}} \Big\{\sum_{j\in V_k^b} t_j - \delta\big|\widetilde{V}_k^b\big|\Big\}\Big\} \\ &= m^b \cdot \max\Big\{c(b,t), \max_{k\in K(b,t)\setminus\{f\}} \big\{c(b,t) - \delta\big|\widetilde{V}_k^b\big|\big\}\Big\} = m^b \cdot c(b,t) \underbrace{=}_{\text{Property 1}} m^{b^0} \cdot c(b^0,t) \\ &= m^{b^0} \cdot \max_{k\in K(b^0,t)} \sum_{j\in V_k^{b^0}} t_j \underbrace{>}_{\widetilde{V}_k^{b^0}\neq\emptyset,\ k\in K(b^0,t)} m^{b^0} \cdot \max_{k\in K(b^0,t)} \Big\{\sum_{j\in V_k^{b^0}} t_j - \delta\big|\widetilde{V}_k^{b^0}\big|\Big\} \\ &= m^{b^0} \cdot \max_{k\in K(b^0,t)} \sum_{j\in V_k^{b^0}} t_j^* \underbrace{=}_{\text{Property 5}} m^{b^0} \cdot c(b^0,t^*). \end{aligned}$$

Case 2: $\widetilde{V}_f^b \neq \emptyset$. Then for an arbitrarily chosen task $p \in \widetilde{V}_f^b$ we consider two subcases:

• **Subcase 2.1**: There exists a set $\widetilde{V}_g^{b^0} \in \Upsilon(b^0,t)$, $g \in K(b^0,t)$, such that $p \in \widetilde{V}_g^{b^0}$. Since $\widetilde{V}_f^b \neq \widetilde{V}_g^{b^0}$, two subsubcases are possible:

• **Subsubcase 2.1.1**: There exists $s \in \widetilde{V}_f^b$ such that $s \notin \widetilde{V}_g^{b^0}$. Then, taking

$$t_j^* = \begin{cases} t_j + \delta, & \text{if } j \in \{p,s\}, \\ t_j & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon$, we obtain that $t^* \in \Omega(\varepsilon, t)$. Using Property 1 and Remark 3, we obtain

$$m^b \cdot c(b, t^*) = m^b \cdot \max_{i \in W^b} \sum_{j \in V_i^b} t_j^* = m^b \cdot \max\left\{ \sum_{j \in V_f^b} t_j^*, \max_{i \in W^b \setminus \{f\}} \sum_{j \in V_i^b} t_j^* \right\}$$

$$= m^b \cdot \max\left\{ \sum_{j \in V_f^b} t_j + 2\delta, \max_{i \in W^b \setminus \{f\}} \sum_{j \in V_i^b} t_j \right\} \underbrace{=}_{f \in K(b,t)} m^b \cdot \big(c(b, t) + 2\delta\big)$$

$$\underbrace{=}_{\text{Property 1}} m^{b^0} \cdot c\big(b^0, t\big) + 2m^b\delta \underbrace{>}_{\text{Remark 3}} m^{b^0} \cdot c\big(b^0, t\big) + m^{b^0}\delta$$

$$\underbrace{=}_{g \in K(b^0,t)} m^{b^0} \cdot \max\left\{ \sum_{j \in V_g^{b^0}} t_j + \delta, \max_{i \in W^{b^0} \setminus \{g\}} \sum_{j \in V_i^{b^0}} t_j^* \right\}$$

$$= m^{b^0} \cdot \max\left\{ \sum_{j \in V_g^{b^0}} t_j^*, \max_{i \in W^{b^0} \setminus \{g\}} \sum_{j \in V_i^{b^0}} t_j^* \right\} = m^{b^0} \cdot c\big(b^0, t^*\big).$$

- **Subsubcase 2.1.2**: There exists $s \in \widetilde{V}_g^{b^0}$ such that $s \notin \widetilde{V}_f^b$. Then, taking

$$t_j^* = \begin{cases} t_j + \delta, & \text{if } i = p, \\ t_j - \delta, & \text{if } i = s, \\ t_j & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon$, we derive that $t^* \in \Omega(\varepsilon, t)$ and, using Property 1, we have

$$m^b \cdot c(b, t^*) = m^b \cdot \max_{i \in W^b} \sum_{j \in V_i^b} t_j^* = m^b \cdot \max\left\{ \sum_{j \in V_f^b} t_j^*, \max_{i \in W^b \setminus \{f\}} \sum_{j \in V_i^b} t_j^* \right\}$$

$$= m^b \cdot \max\left\{ \sum_{j \in V_f^b} t_j + \delta, \max_{i \in W^b \setminus \{f\}} \sum_{j \in V_i^b} t_j^* \right\} \underbrace{=}_{f \in K(b,t)} m^b \cdot c(b, t) + m^b\delta$$

$$\underbrace{=}_{\text{Property 1}} m^{b^0} \cdot c\big(b^0, t\big) + m^b\delta > m^{b^0} \cdot c\big(b^0, t\big) = m^{b^0} \cdot \max\left\{ \sum_{j \in V_g^{b^0}} t_j, \max_{i \in W^{b^0} \setminus \{g\}} \sum_{j \in V_i^{b^0}} t_j \right\}$$

$$= m^{b^0} \cdot \max\left\{ \sum_{j \in V_g^{b^0}} t_j^*, \max_{i \in W^{b^0} \setminus \{g\}} \sum_{j \in V_i^{b^0}} t_j^* \right\} = m^{b^0} \cdot c\big(b^0, t^*\big).$$

- **Subcase 2.2**: No set from $\Upsilon(b^0, t)$ contains operation p. Then, taking

$$t_j^* = \begin{cases} t_j + \delta, & \text{if } j = p, \\ t_j & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon$, we obtain that $t^* \in \Omega(\varepsilon, t)$ and, using Property 1, we derive the same results as in Subsubcase 2.1.2.

Resuming all the cases described above, we conclude that

$$\forall \varepsilon < \min\{\Delta(b), \Delta(b^0)\}\ \exists t^* \in \Omega(\varepsilon, t) \quad \big(\mathcal{Z}(b, t^*) > \mathcal{Z}(b^0, t^*)\big),$$

i.e.

$$\forall \varepsilon < \min\{\Delta(b), \Delta(b^0)\}\ \exists t^* \in \Omega(\varepsilon, t) \quad \big(b \notin \mathbf{B}_{\mathcal{O}}(t^*)\big).$$

From this $\Xi_{\mathcal{O}}(b, t) = \emptyset$ and, according to Definition 2, b is not $\mathcal{O}$-stable. □

Lemma 2 *If for* $b, b^0 \in \mathbf{B}_{\mathcal{O}}(t)$ *inclusion* $\Upsilon(b,t) \subseteq \Upsilon(b^0,t)$ *and equality* $m^b = m^{b^0}$ *hold, then the following is true*:

$$\forall \varepsilon < \min\{\Delta(b), \Delta(b^0)\} \quad \forall t' \in \Omega(\varepsilon, t) \quad \big(\mathcal{Z}(b,t') \le \mathcal{Z}(b^0,t')\big). \tag{10}$$

Proof For any $t' \in \Omega(\varepsilon, t)$, where $\varepsilon < \min\{\Delta(b), \Delta(b^0)\}$, taking into account Properties 1 and 5 and the representation of variations of uncertain tasks, we derive

$$\begin{aligned} c(b,t') &\underbrace{=}_{\text{Property 5}} \max_{k \in K(b,t)} \sum_{j \in V_k^b} t'_j = \max_{k \in K(b,t)} \Big\{ \sum_{j \in V_k^b} t_j + \sum_{j \in \widetilde{V}_k^b} \xi_j \Big\} \\ &= \max_{U \in \Upsilon(b,t)} \Big\{ c(b,t) + \sum_{j \in U} \xi_j \Big\} \underbrace{=}_{\text{Property 1}} \max_{U \in \Upsilon(b,t)} \Big\{ c(b^0,t) + \sum_{j \in U} \xi_j \Big\} \\ &\underbrace{\le}_{\Upsilon(b,t) \subseteq \Upsilon(b^0,t)} \max_{U \in \Upsilon(b^0,t)} \Big\{ c(b^0,t) + \sum_{j \in U} \xi_j \Big\} = \max_{k \in K(b^0,t)} \sum_{j \in V_k^{b^0}} t'_j \underbrace{=}_{\text{Property 5}} c(b^0,t'). \end{aligned}$$

This implies (10). □

Lemma 3 *Let* $b, b^0 \in \mathbf{B}_{\mathcal{O}}(t)$, $\Upsilon(b,t) \subseteq \Upsilon(b^0,t)$ *and* $m^b \neq m^{b^0}$. *Then the following is true*:

- *if* $\emptyset \notin \Upsilon(b^0,t)$, *then* b *is not* $\mathcal{O}$*-stable*;
- *if* $\emptyset \in \Upsilon(b^0,t)$, *then*
  - *if* $\Upsilon(b,t) \neq \{\emptyset\}$ *and* $m^b > m^{b^0}$, *then* b *is not* $\mathcal{O}$*-stable*;
  - *otherwise formula* (10) *holds.*

Proof Let us consider two following cases.

Case 1: $\emptyset \notin \Upsilon(b^0,t)$. Taking into account inclusion $\Upsilon(b,t) \subseteq \Upsilon(b^0,t)$, we conclude that $\emptyset \notin \Upsilon(b,t)$ and since $\Upsilon(b,t)$ and $\Upsilon(b^0,t)$ are not empty sets (see the definition of $\Upsilon(b,t)$), then they consist of non-empty sets of uncertain operations.

• **Subcase 1.1**: $m^b > m^{b^0}$. Following the conclusions of Case 1, there exist an index $f \in K(b,t)$ such that $\widetilde{V}_f^b \neq \emptyset$ and $g \in K(b^0,t)$ such that $\widetilde{V}_f^b = \widetilde{V}_g^{b^0}$. Using this fact, we set

$$t_j^* = \begin{cases} t_j + \delta, & \text{if } j \in \widetilde{V}_f^b, \\ t_j & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon$. Therefore, $t^* \in \Omega(\varepsilon, t)$ and due to $m^b > m^{b^0}$ and Property 1 we obtain

$$\begin{aligned} m^b \cdot c(b,t^*) &= m^b \cdot \max_{i \in W^b} \sum_{j \in V_i^b} t_j^* = m^b \cdot \max\Big\{ \sum_{j \in V_f^b} t_j^*, \max_{i \in W^b \setminus \{f\}} \sum_{j \in V_i^b} t_j^* \Big\} \\ &= m^b \cdot \max\Big\{ \sum_{j \in V_f^b} t_j + \delta \big|\widetilde{V}_f^b\big|, \max_{i \in W^b \setminus \{f\}} \sum_{j \in V_i^b} t_j \Big\} \underbrace{=}_{f \in K(b,t)} m^b \cdot c(b,t) + m^b \delta \big|\widetilde{V}_f^b\big| \\ &\underbrace{=}_{\text{Property 1},\ \widetilde{V}_f^b = \widetilde{V}_g^{b^0}} m^{b^0} \cdot c(b^0,t) + m^b \delta \big|\widetilde{V}_g^{b^0}\big| \underbrace{>}_{m^b > m^{b^0}} m^{b^0} \cdot c(b^0,t) + m^{b^0} \delta \big|\widetilde{V}_g^{b^0}\big| \\ &= m^{b^0} \cdot \max\Big\{ \sum_{j \in V_g^{b^0}} t_j^*, \max_{i \in W^{b^0} \setminus \{g\}} \sum_{j \in V_i^{b^0}} t_j^* \Big\} = m^{b^0} \cdot c(b^0,t^*). \end{aligned}$$

• **Subcase 1.2**: $m^b < m^{b^0}$. Setting

$$t_j^* = \begin{cases} t_j - \frac{\delta}{|\widetilde{V}_k^{b^0}|}, & \text{if } j \in \widetilde{V}_k^{b^0}, k \in K(b^0, t), \\ t_j & \text{otherwise,} \end{cases}$$

where $0 < \delta < \varepsilon < \min\{\Delta(b), \Delta(b^0)\}$, we get that $t^* \in \Omega(\varepsilon, t)$ and, taking into account Property 5, we derive

$$\begin{aligned} c\big(b, t^*\big) &= \max_{k \in K(b,t)} \sum_{j \in V_k^b} t_j^* = \max_{k \in K(b,t)} \Big\{ \sum_{j \in V_k^b \setminus \widetilde{V}_k^b} t_j + \sum_{j \in \widetilde{V}_k^b} t_j^* \Big\} \\ &= \max_{k \in K(b,t)} \Big\{ \sum_{j \in V_k^b \setminus \widetilde{V}_k^b} t_j + \sum_{j \in \widetilde{V}_k^b} \Big(t_j - \frac{\delta}{|\widetilde{V}_k^b|} \Big) \Big\} = \max_{k \in K(b,t)} \Big\{ \sum_{j \in V_k^b} t_j - \delta \Big\} = c(b, t) - \delta. \end{aligned}$$

In the same way we obtain $c(b^0, t^*) = c(b^0, t) - \delta$. Therefore, due to Property 1 we have the following:

$$\begin{aligned} m^b \cdot c\big(b, t^*\big) &= m^b \cdot c(b, t) - m^b \delta = m^{b^0} \cdot c\big(b^0, t\big) - m^b \delta \\ &> m^{b^0} \cdot c\big(b^0, t\big) - m^{b^0} \delta = m^{b^0} \cdot c\big(b^0, t^*\big). \end{aligned}$$

Resuming these two subcases, we conclude

$$\forall \varepsilon < \min\big\{\Delta(b), \Delta\big(b^0\big)\big\}\ \exists t^* \in \Omega(\varepsilon, t) \quad \big(\mathcal{Z}\big(b^0, t^*\big) < \mathcal{Z}\big(b, t^*\big)\big). \tag{11}$$

In other words, $\Xi_{\mathcal{O}}(b, t) = \emptyset$ and, according to Definition 2, b is not $\mathcal{O}$-stable.

Case 2: $\emptyset \in \Upsilon(b^0, t)$. Let us consider two subcases.

• **Subcase 2.1**: $\Upsilon(b, t) \neq \{\emptyset\}$ and $m^b > m^{b^0}$. Since $\Upsilon(b, t) \neq \{\emptyset\}$, then due to inclusion $\Upsilon(b, t) \subseteq \Upsilon(b^0, t)$ there exist indices $f \in K(b, t)$ and $g \in K(b^0, t)$ such that $\widetilde{V}_f^b = \widetilde{V}_g^{b^0} \neq \emptyset$. Therefore, repeating the same calculations as in Subcase 1.1, we derive the formula (11), i.e. b is not $\mathcal{O}$-stable due to Definition 2.

• **Subcase 2.2**: $\Upsilon(b, t) = \{\emptyset\}$ or $m^b < m^{b^0}$. Then, according to Properties 1, 5 and the representation of perturbations of uncertain tasks, for any $t' \in \Omega(\varepsilon, t)$, where $\varepsilon < \min\{\Delta(b), \Delta(b^0)\}$, we derive:

$$\begin{aligned} m^b \cdot c\big(b, t'\big) &\underbrace{=}_{\text{Property 5}} m^b \cdot \max_{k \in K(b,t)} \sum_{j \in V_k^b} t'_j = m^b \cdot \max_{k \in K(b,t)} \Big\{ \sum_{j \in V_k^b} t_j + \sum_{j \in \widetilde{V}_k^b} \xi_j \Big\} \\ &= \max_{U \in \Upsilon(b,t)} \Big\{ m^b \cdot c(b, t) + m^b \cdot \sum_{j \in U} \xi_j \Big\} \\ &\underbrace{=}_{\text{Property 1}} \max_{U \in \Upsilon(b,t)} \Big\{ m^{b^0} \cdot c\big(b^0, t\big) + m^b \cdot \sum_{j \in U} \xi_j \Big\} \\ &= m^{b^0} \cdot c\big(b^0, t\big) + m^b \cdot \max_{U \in \Upsilon(b,t)} \sum_{j \in U} \xi_j = \mathcal{I}. \end{aligned}$$

If $\Upsilon(b, t) = \{\emptyset\}$, then $\max_{U \in \Upsilon(b,t)} \sum_{j \in U} \xi_j = 0$ and we obtain:

$$\begin{aligned} \mathcal{I} &= m^{b^0} \cdot c\big(b^0, t\big) + m^{b^0} \cdot \max_{U \in \Upsilon(b,t)} \sum_{j \in U} \xi_j \\ &\underbrace{\leq}_{\Upsilon(b,t) \subseteq \Upsilon(b^0,t)} m^{b^0} \cdot c\big(b^0, t\big) + m^{b^0} \cdot \max_{U \in \Upsilon(b^0,t)} \sum_{j \in U} \xi_j \end{aligned}$$

$$= m^{b^0} \cdot \max_{U \in \Upsilon(b^0,t)} \left\{ c\left(b^0, t\right) + \sum_{j \in U} \xi_j \right\} = m^{b^0} \cdot c\left(b^0, t'\right).$$

If $\Upsilon(b,t) \neq \{\emptyset\}$, then $m^b < m^{b^0}$ and we have:

$$\begin{aligned} \mathcal{I} \underbrace{\leq}_{\Upsilon(b,t) \subseteq \Upsilon(b^0,t)} & m^{b^0} \cdot c\left(b^0, t\right) + m^b \cdot \max_{U \in \Upsilon(b^0,t)} \sum_{j \in U} \xi_j \\ & \times \left[\text{Since } \emptyset \in \Upsilon\left(b^0, t\right), \text{ then } \max_{U \in \Upsilon(b^0,t)} \sum_{j \in U} \xi_j \geq 0 \right] \\ \underbrace{\leq}_{m^b < m^{b^0}} & m^{b^0} \cdot c\left(b^0, t\right) + m^{b^0} \cdot \max_{U \in \Upsilon(b^0,t)} \sum_{j \in U} \xi_j \\ = & m^{b^0} \cdot \max_{U \in \Upsilon(b^0,t)} \left\{ c\left(b^0, t\right) + \sum_{j \in U} \xi_j \right\} = m^{b^0} \cdot c\left(b^0, t'\right). \end{aligned}$$

In other words, we have formula (10). □

The results of Lemmas 1–3 and Corollaries 1 and 3 imply

Theorem 3 *Balance $b \in \mathbf{B}_{\mathcal{O}}(t)$ is not $\mathcal{O}$-stable iff at least one of two following conditions holds*:

- *there exists $k \in \widetilde{W}^b$ such that $c = t(V_k^b)$*,
- *there exists $b^0 \in \mathbf{B}_{\mathcal{O}}(t)$ such that the following is true*:
  $\Upsilon(b,t) \subseteq \Upsilon(b^0,t) \Rightarrow (m^b \neq m^{b^0} \,\&\, (\emptyset \in \Upsilon(b^0,t) \Rightarrow (\Upsilon(b,t) \neq \{\emptyset\} \,\&\, m^b > m^{b^0})))$.

For two mentioned above partial cases of SALBP-E, namely SALBP-1 and SALBP-2 that can be expressed as follows:
SALBP-1: $\mathcal{Z}(b,t) := m^b \to \min_{b \in \mathbf{B}_{\mathcal{F}}(t)}$, $c(b,t) \leq c$, m is not given,
SALBP-2: $\mathcal{Z}(b,t) := c(b,t) \to \min_{b \in \mathbf{B}_{\mathcal{F}}(t)}$, c is not given, $m^b = m$,
Theorem 3 implies

Corollary 4 (Sotskov et al. 2006) *Balance $b \in \mathbf{B}_{\mathcal{O}}(t)$ of SALBP-1 is not $\mathcal{O}$-stable iff there exists $k \in \widetilde{W}^b$ such that $c = t(V_k^b)$.*

Corollary 5 *Balance $b \in \mathbf{B}_{\mathcal{O}}(t)$ of SALBP-2 is not $\mathcal{O}$-stable iff there exists $b^0 \in \mathbf{B}_{\mathcal{O}}(t)$ such that $\Upsilon(b,t) \subseteq \Upsilon(b^0,t)$ does not hold.*

Denote by $\mathcal{O}$-Stab(b,t) the following decision problem: Is balance $b \in \mathbf{B}_{\mathcal{O}}(t)$, $\mathcal{O}$-stable? Analyzing Theorem 3 and Corollary 5, it can be concluded that $\mathcal{O}$-Stab(b,t) is a difficult computing problem for SALBP-E and SALBP-2, since it is mandatory to know the whole set of optimal balances. As a consequence, finding the corresponding $\mathcal{O}$-stability radius is a more difficult problem. However, a useful upper bound of the $\mathcal{O}$-stability radius of balance $b \in \mathbf{B}_{\mathcal{O}}(t)$ can be polynomially calculated for SALBP-E due to the inequality $\rho_{\mathcal{O}}(b,t) \leq \rho_{\mathcal{F}}(b,t)$ and Theorem 1.

Corollary 6 *For $\mathcal{O}$-stability radius $\rho_{\mathcal{O}}(b,t)$ of balance $b \in \mathbf{B}_{\mathcal{O}}(b,t)$ of SALBP-E the following inequality holds*:

$$\rho_{\mathcal{O}}(b,t) \leq \min_{k \in \widetilde{W}^b} \frac{c - t(V_k^b)}{|\widetilde{V}_k^b|}.$$

At the same time, problem $\mathcal{O}$-Stab(b,t) is polynomially solvable for SALBP-1 due to Corollary 4. An algorithm similar to that presented in Sect. 4 can be applied.

Let us remark that since for SALBP-1 the working time of workstations is only limited by a fixed value c and relations between them for different balances are not taken into account, then in view of the reasonings presented at the beginning of this section, we conclude that an optimal balance of SALBP-1 preserves its optimality under VTPT iff it remains feasible and no quasi-feasible balance with less number of workstations becomes feasible. From the latter proposition, Theorems 1 and 2 imply

Corollary 7 (Sotskov et al. 2006) *$\mathcal{O}$-stability radius $\rho_{\mathcal{O}}(b,t)$ of balance $b \in \mathbf{B}_{\mathcal{O}}(t)$ for SALBP-1 is calculated as follows*:

$$\rho_{\mathcal{O}}(b,t) = \min\Big\{\rho_{\mathcal{F}}(b,t), \min_{b' \in \mathbf{B}_{\widehat{\mathcal{F}}}(t)} \big\{\rho_{\widehat{\mathcal{F}}}\big(b',t\big) : m^{b'} < m^{b}\big\}\Big\}.$$

Nevertheless, it follows that finding $\mathcal{O}$-stability radius $\rho_{\mathcal{O}}(b,t)$ for SALBP-1 is also a difficult problem, since it requires to know the whole set of quasi-feasible balances having less workstations than for balance b.

7 Finding a compromise

It is clear that there is not a feasible balance with the minimal objective function value and the maximal value of the $\mathcal{F}$-stability radius at the same time. Therefore, in presence of VTPT finding a compromise between these two antagonistic aims (minimizing the objective function and maximizing the $\mathcal{F}$-stability radius) is an important reasonable issue.

The concept of Pareto-optimality (Ehrgott 2005) is used in this paper as a compromise. This means that the balances in which it is impossible to improve one of the aims without making worse another one are sought for. Such balances are called efficient or non-dominated.

To formally present set $\mathcal{NDB}$ of non-dominated balances, the binary relation between any two feasible balances b and b' reflecting the Pareto dominant rule is introduced as follows:

$$b \succ b' \quad \Longleftrightarrow \quad \mathcal{Z}(b,t) \le \mathcal{Z}\big(b',t\big) \ \&\ \rho_{\mathcal{F}}(b,t) \ge \rho_{\mathcal{F}}\big(b',t\big),$$

where strict inequality holds at least once. In the case where $b \succ b'$ we say that b dominates b' or b' is dominated by b.

Thus, given set $\mathcal{B}$ of known (found) feasible balances, we have

$$\mathcal{NDB} = \big\{b \in \mathcal{B} : \nexists b' \in \mathcal{B} \ \ \big(b' \succ b\big)\big\}.$$

The goal of this section is to propose a construction approach of $\mathcal{NDB}$.

7.1 Description of a suggested approach

It is quite natural that finding set $\mathcal{NDB}$ is a difficult problem. Therefore, developing construction methods of appropriate approximations of $\mathcal{NDB}$ seems a justified choice.

To do this, a multi-start heuristic procedure is used. Each iteration of this procedure consists of two phases: constructing a feasible balance (using heuristic $\mathcal{H}(c)$ described in Sect. 7.1.1) and comparing it to these already known. Both phases are interchangeably repeated until a stopping criterion is satisfied.

At the beginning of the heuristic procedure, an integer interval $[c_{\min}, c_{\max}]$ is defined determining the set of $c_{\max} - c_{\min} + 1$ admissible values for c. These values are sequentially used at the construction phase. The value of c is originally initiated as $c_{\max}$ and decreased by one each $\frac{T_{\max}}{c_{\max}-c_{\min}+1}$ time-period, where $T_{\max}$ is the available solution time.

7.1.1 *Heuristic $\mathcal{H}(c)$*

Given a current value of c, this heuristic constructs a feasible balance by assigning as many tasks as possible to the current workstation. At the beginning, the current feasible balance contains only one empty workstation. The heuristic assigns tasks to it until no tasks can be added because of existing constraints. Then a new empty workstation is opened which becomes current and the heuristic assigns tasks to it.

This continues until either all tasks are assigned and a feasible balance is obtained or it is impossible to open a new workstation without exceeding m. In the latter case, the balance is considered to be unfeasible. Therefore, it is excluded from consideration and a new one is constructed.

Let b and k be the constructed balance and the index of the current workstation in it, respectively. And let V_k^b be the set of tasks assigned to workstation k of balance b. To choose an operation to be assigned to workstation k, the so-called Candidate List $\mathcal{CL}(k, c)$ of tasks is generated. List $\mathcal{CL}(k, c)$ contains all tasks that can be assigned to workstation k. This list is built in the following way: set of unassigned tasks is looked through and task j is added to $\mathcal{CL}(k, c)$ if all following conditions are satisfied:

- all predecessors of j have been already assigned;
- $t(V_k^b \cup \{j\}) \le c$.

If $\mathcal{CL}(k, c) = \emptyset$, no more tasks can be assigned to the current workstation. A new workstation is opened if it possible and $\mathcal{CL}(k + 1, c)$ is built. Otherwise a task j is randomly chosen from $\mathcal{CL}(k, c)$ and assigned to V_k^b, $\mathcal{CL}(k, c)$ is rebuilt.

It is clear that less the quantity of uncertain tasks at a workstation, greater the $\mathcal{F}$-stability radius is. Thereby, it is required to assign uncertain tasks together at the same workstation as little as possible. To do this, a task j can be randomly chosen from $\mathcal{CL}(k, c)$ only in the case where uncertain tasks have not been assigned yet to the current workstation ($\widetilde{V}_k^b = \emptyset$) or $\mathcal{CL}(k, c)$ is composed of only uncertain tasks ($\mathcal{CL}(k, c) \cap \widetilde{V} = \mathcal{CL}(k, c)$), otherwise j is chosen from $\mathcal{CL}(k, c) \setminus \widetilde{V}$.

To distinguish these two strategies, the notation $\mathcal{H}_1(c)$ is used for the basic version of $\mathcal{H}(c)$ heuristic and $\mathcal{H}_2(c)$ for the second one.

7.1.2 *Construction of $\mathcal{NDB}$*

A consecutive construction of $\mathcal{NDB}$ is used. At the beginning, sets $\mathcal{B}$ and $\mathcal{NDB}$ are empty sets, where $\mathcal{B}$ sequentially accumulates all feasible balances constructed by heuristic $\mathcal{H}(c)$, while non-dominated ones constitute $\mathcal{NDB}$ which is regenerated each time when a new feasible balance is obtained.

The analyze of a new found feasible balance b is based on the following three propositions which are valid due to the transitivity of $\succ$ and internal and external stability of $\mathcal{NDB}$ (Ehrgott 2005).

Proposition 1 *If there is $b' \in \mathcal{B}$ such that $b' \succ b$, then*

- *there is $b'' \in \mathcal{NDB}$ such that $b'' \succ b$;*
- *no balance from $\mathcal{NDB}$ is dominated by b.*

Table 1 Benchmark tests

Test name	n	t_{min}	t_{max}	t_{sum}	c_{min}	c_{max}	m	OS	T_{max}
Buxey	29	1	25	324	27	54	13	50.74	300
Gunther	35	1	40	483	41	81	14	59.5	300
Lutz2	89	1	10	485	11	21	49	77.55	300
Mitchell	21	1	13	105	14	39	8	70.95	300
Roszieg	25	1	13	125	14	32	10	71.67	300
Sawyer	30	1	25	324	25	75	14	44.83	300
Wee-Mag	75	2	27	1499	28	56	63	22.67	300
Barthol2	148	1	83	4234	84	170	51	25.8	600
Lutz3	89	1	74	1644	75	150	23	77.55	600
Warnecke	58	7	53	1548	54	111	31	59.1	600
Barthold	148	3	383	5634	403	805	14	25.8	900
Heskia	28	1	108	1024	138	342	8	22.49	900
Kilbridge	45	3	55	552	56	184	10	44.55	900
Mukherje	94	8	171	4208	176	351	25	44.8	900
Tonge	70	1	156	3510	160	572	23	59.42	900

Proposition 2 *If there is $b' \in \mathcal{NDB}$ such that $b \succ b'$, then $b \in \mathcal{NDB}$.*

It is easy to see that if the condition of Proposition 2 holds, then balance b' is excluded from set $\mathcal{NDB}$.

Proposition 3 *If no balance from $\mathcal{NDB}$ dominates b, then $b \in \mathcal{NDB}$.*

Propositions 1, 2 and 3 conclude that it is required to compare a new constructed feasible balance b only with the current balances from $\mathcal{NDB}$, since if b is dominated, it is necessary dominated by one of the balances from $\mathcal{NDB}$, otherwise b belongs to $\mathcal{NDB}$.

Thus, given a new constructed feasible balance b, the following sequence of operations is done. If $\mathcal{NDB} = \emptyset$, then b is the first constructed feasible balance and therefore it is added to $\mathcal{NDB}$. If not, two cases are possible: there is a balance $b' \in \mathcal{NDB}$ such that $b' \succ b$, then $\mathcal{NDB}$ remains unchangeable due to Proposition 1; otherwise all balances from $\mathcal{NDB}$ dominated by b are excluded and b is added to $\mathcal{NDB}$ in view of Propositions 2 and 3.

8 Computational experiments

8.1 Experimental conditions

The experiments were carried out on Pentium-IV (3 GHz, 2.5 GB RAM). The suggested approaches have been implemented in C++ and evaluated on 15 well-known benchmark tests of SALBPs (see http://www.assembly-line-balancing.de) presented in Table 1.

Besides the notations introduced above, Table 1 also gives

- $t_{min} = \min\{t_j : j \in V\}$ is the minimal task processing time;
- $t_{max} = \max\{t_j : j \in V\}$ is the maximal task processing time;

Table 2 First approach

Test name	$\#\mathcal{NDB}_1$	$\mathcal{Z}_{\min}$	$\mathcal{Z}_{av}$	$\mathcal{Z}_{\max}$	$\rho_{\min}$	ρ_{av}	$\rho_{\max}$
Buxey	5	282	304.2	343	2	3.8	5.67
Gunther	12	432	482	539	0	3.72	7.5
Lutz2	4	483	513.75	540	0	0.54	1
Mitchell	3	70	90.33	105	2.5	3.17	4
Roszieg	6	96	109.33	119	0	0.92	2
Sawyer	5	256	276.8	329	2.5	4.3	6.5
Wee-Mag	9	1504	1659.33	1952	0	3.17	5.5
Barthol2	7	4160	4301.14	4592	0	1.1	2
Lutz3	9	1540	1657.89	1890	0	1.63	2.75
Warnecke	6	1485	1537.17	1648	0.5	3.56	7
Barthold	9	4935	5160	5628	0.58	1.21	1.6
Heskia	9	682	843.11	1012	0.5	6.86	10
Kilbridge	7	368	452.71	534	0	3	4
Mukherje	8	3924	4096.13	4466	0	2.08	3.4
Tonge	10	3012	3166.5	3549	3	6.18	9

- $t_{sum} = \sum_{j=1}^{n} t_j$ is the sum of processing times of the given tasks;
- $OS = \frac{2|\mathcal{A}|}{|V|(|V|-1)}$ is the order strength of the precedence constraints represented by graph $G = (V, \mathcal{A})$.

For each test, 33 % of tasks were chosen at random as uncertain. Available computational time $T_{\max}$ was limited by 300 s for tests where $c_{\max} - c_{\min} \le 50$; by 600 s for tests where $50 < c_{\max} - c_{\min} \le 100$; and by 900 s for the rest.

8.2 Analysis of the obtained results

Tables 2 and 3 show respectively the performance of the considered approaches (first of them using $\mathcal{H}_1(c)$ heuristic and the second one $\mathcal{H}_2(c)$), where the following notations are used: # is the cardinality of the corresponding set, $\mathcal{NDB}_1$ and $\mathcal{NDB}_2$ are the sets of the non-dominated balances found by the first and the second approaches, respectively; $\mathcal{Z}_{\min}$, $\mathcal{Z}_{av}$, and $\mathcal{Z}_{\max}$ are respectively the minimal, average, and maximal values of their objective function; $\rho_{\min}$, ρ_{av}, and $\rho_{\max}$ are respectively the minimal, average, and maximal values of their $\mathcal{F}$-stability radius.

Table 4 presents the comparison between two developed approaches. Here $\mathcal{NDB}_{mut}$ is the set of the non-dominated balances found mutually by two approaches. In other words,

$$\mathcal{NDB}_{mut} = \{b \in \mathcal{NDB}_1 \cup \mathcal{NDB}_2 : \nexists b' \in \mathcal{NDB}_1 \cup \mathcal{NDB}_2 \ (b' \succ b)\}.$$

$\mathcal{B}_{mut\text{-}1}$ and $\mathcal{B}_{mut\text{-}2}$ are the sets of the balances from $\mathcal{NDB}_{mut}$ found only by the first or the second approaches, respectively, i.e. $\mathcal{B}_{mut\text{-}1} = \mathcal{NDB}_{mut} \cap \mathcal{NDB}_1$ and $\mathcal{B}_{mut\text{-}2} = \mathcal{NDB}_{mut} \cap \mathcal{NDB}_2$. Θ_1 and Θ_2 are the percents of balances from $\mathcal{NDB}_1$ and $\mathcal{NDB}_2$, respectively, which belong to $\mathcal{NDB}_{mut}$, i.e. $\Theta_1 = \frac{\#\mathcal{B}_{mut\text{-}1}}{\#\mathcal{NDB}_1} \cdot 100\ \%$ and $\Theta_2 = \frac{\#\mathcal{B}_{mut\text{-}2}}{\#\mathcal{NDB}_2} \cdot 100\ \%$.

Analyzing the obtained results, it can be concluded that generally the first approach provided better results than the second one. For example, the first approach found on average

Table 3 Second approach

Test name	$\#\mathcal{NDB}_2$	$\mathcal{Z}_{min}$	$\mathcal{Z}_{av}$	$\mathcal{Z}_{max}$	ρ_{min}	ρ_{av}	ρ_{max}
Buxey	3	287	311.67	328	3	4	5
Gunther	11	432	483.91	560	0	3.44	7
Lutz2	3	483	517.67	570	0	0.5	1
Mitchell	3	70	90.33	105	2.5	3.17	4
Roszieg	6	96	109.33	119	0	0.92	2
Sawyer	5	256	284.6	318	2.5	4.6	6
Wee-Mag	6	1519	1612	1705	1	3.41	5
Barthol2	7	4150	4220.43	4292	0.33	1.05	2
Lutz3	4	1548	1569.75	1599	0	1.27	2
Warnecke	9	1485	1546.11	1648	1.5	4.27	7
Barthold	7	4935	4989.43	5120	0.3	0.96	1.67
Heskia	11	682	902.73	1080	0.33	5.92	8.88
Kilbridge	6	368	461.17	621	0	2.77	4
Mukherje	14	3924	4076.36	4410	0.43	3.02	5.75
Tonge	9	3018	3473.88	4202	1.66	7.1	10.67

Table 4 Approaches comparison

Test name	$\#\mathcal{NDB}_{mut}$	$\#\mathcal{B}_{mut\text{-}1}$	Θ_1, %	$\#\mathcal{B}_{mut\text{-}2}$	Θ_2, %
Buxey	5	5	100	1	33.33
Gunther	13	12	100	4	36.36
Lutz2	4	4	100	2	66.67
Mitchell	3	3	100	3	100
Roszieg	6	6	100	6	100
Sawyer	7	5	100	5	100
Wee-Mag	8	5	55.56	5	83.33
Barthol2	7	1	14.29	7	100
Lutz3	10	8	88.89	2	50
Warnecke	9	2	33.33	8	88.88
Barthold	5	2	22.22	3	42.86
Heskia	9	9	100	0	0
Kilbridge	7	6	85.71	2	33.33
Mukherje	14	1	12.5	14	100
Tonge	10	5	50	5	55.56

7.27 non-dominated balances, where 70.83 % of them participate in $\mathcal{NDB}_{mut}$, while the second one found on average 6.93 non-dominated balances, where 66.02 % from them take part in $\mathcal{NDB}_{mut}$.

However, for three tested benchmarks the second approach provided substantially better balances than the first one (see Barthol2's, Warnecke's, and Mukherje's tests in Table 4).

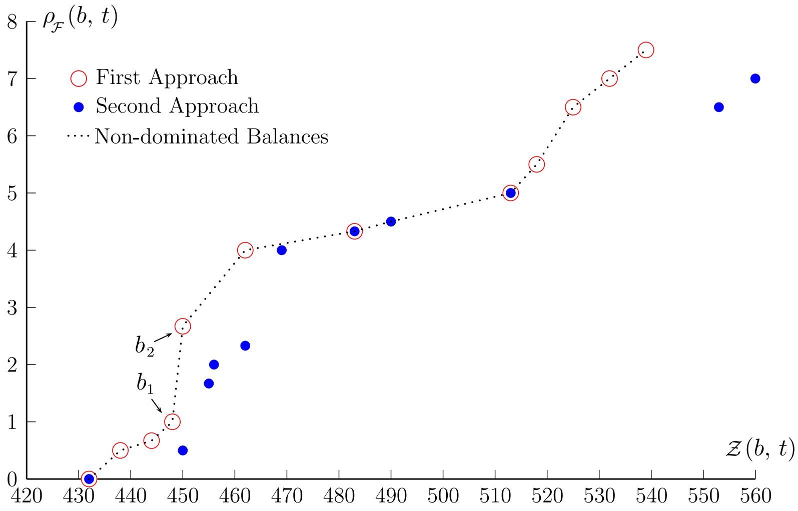


Fig. 3 Comparison of two approaches for Gunther's test

And, as expected, although the second approach found on average less number of non-dominated balances than the first one, it provided better results with respect to the $\mathcal{F}$-stability radius, since the average value of the $\mathcal{F}$-stability radius is equal to 3.09 for the second approach, whereas it is 3.02 for the first one.

The complementary character of these two approaches for several tests should be also noted. For instance, for Barthold's and Tonge's tests each approach found approximately a half of $\mathcal{NDB}_{\text{mut}}$ without any common balances. Similarly, for Sawyer's and Wee-Mag's tests, the same quantity of balances constituting set $\mathcal{NDB}_{\text{mut}}$ was found by each of the approaches, but with a sufficiently small percent of common balances.

An interesting result was obtained for Mitchell's and Roszieg's tests, where exactly the same set $\mathcal{NDB}$ was found by two approaches and, as a consequence, all found balances constitute set $\mathcal{NDB}_{\text{mut}}$.

The performance of two proposed approaches for Gunther's test is shown in Fig. 3. For a designer looking for a line configuration with the efficiency from $[445, 460]$, between two balances b^1 and b^2 such that $\mathcal{Z}(b^1, t) = 448$, $\rho_{\mathcal{F}}(b^1, t) = 1$ and $\mathcal{Z}(b^2, t) = 450$, $\rho_{\mathcal{F}}(b^2, t) = 2.67$, the second one is more preferable, since its value of the $\mathcal{F}$-stability radius is considerably greater than for the first one, in spite of an insignificant disbenefit with respect to its efficiency.

9 Conclusions

In this paper, Simple Assembly Line Balancing Problem of type E (SALBP-E) was studied under variations of task processing times. The tasks' uncertainty was modeled using general concepts of stability analysis, i.e. by small perturbations of their processing time. Conditions of stability, the formula, upper and lower bounds of stability radius for feasible, quasi-feasible, and optimal balances were investigated. The computational complexity of these problems was discussed as well. Polynomial time algorithm computing the stability radius of feasible balances was suggested.

Notice that for SALBP-1 and SALBP-2 similar results was derived here as corollaries from the obtained outcomes of SALBP-E.

The concept of Pareto optimality was used to find a compromise between the objective function to minimize (expressing the efficiency of a feasible balance) and the corresponding stability radius to maximize (reflecting its robustness). Two approaches to find non-dominated balances in terms of Pareto optimality were suggested. These approaches shown a complementary behavior in the numerical experiments on known benchmarks and mutually found on average sufficiently large number of non-dominated balances.

The obtained results are interesting not only theoretically, but can be also useful under preliminary design stage of assembly lines, where designers take into account the variability of processing time for certain tasks and/or dispose only approximate information about the desired efficiency or the robustness level of an assembly line being designed.

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