4OR (2021) 19:93–101
https://doi.org/10.1007/s10288-020-00434-1

RESEARCH PAPER

Min-max controllable risk problems

Evgeny Gurevsky[1] · **Sergey Kovalev**[2] · **Mikhail Y. Kovalyov**[3]

Received: 31 October 2019 / Revised: 21 February 2020 / Published online: 10 March 2020

Abstract
A min-max controllable risk problem, defined on combinatorial structures which are either simple paths of a directed multigraph or spanning trees of an undirected multigraph, with resource dependent risk functions of the arcs or the edges, is studied. The resource amount is limited, and the objective is to distribute it between the arcs or the edges so that the maximum risk over the arcs of a simple path or the edges of a spanning tree is minimized. Two new solution approaches are presented, which are asymptotically faster than the solution approaches suggested in the literature.

Keywords Optimization · Min-max · Risk · Shortest path · Spanning tree · Controllable data

Mathematics Subject Classification 90B10 · 90C27 · 90C32 · 90C47

1 Introduction and literature review

Let E be a set of arbitrary elements, and let $\mathcal{S}(E)$ be a family of non-empty subsets of E, which we call *structures*. Two examples of the sets E and $\mathcal{S}(E)$ are considered. In the first, E is the set of *arcs* of a *directed (multi)graph* $G = (V, E)$, and $\mathcal{S}(E)$ is the set of all *simple paths* between two specified nodes of this graph. In the second, E is the set of *edges* of an *undirected (multi)graph*, for which we keep the same notation $G = (V, E)$, and $\mathcal{S}(E)$ is the set of all *spanning trees* of this graph. We denote cardinalities of the sets V and E as n and m, respectively.

Each element $e \in E$ is associated with a real-valued *resource variable* x_e, lower and upper bounds l_e and u_e on this variable, $l_e < u_e$, and a *risk function* $r_e(x_e) = \frac{u_e - x_e}{u_e - l_e}$. The larger value of x_e implies the lower risk associated with the element e. We call

✉ Evgeny Gurevsky
evgeny.gurevsky@univ-nantes.fr

1 LS2N, Université de Nantes, Nantes, France

2 INSEEC Business School, Lyon, France

3 United Institute of Informatics Problems, National Academy of Sciences of Belarus, Minsk, Belarus

pairs (S, x) *solutions*, where $S \in \mathcal{S}(E)$ is a structure and x is a collection of the variables x_e, $e \in S$. The total resource amount $\sum_{e \in S} x_e$ is upper bounded by a given number B. Input parameters l_e, u_e, $e \in E$, and B are assumed to be non-negative integer numbers. The studied problem is formulated as follows.

Problem Min-Max-Risk:

$$\min_{(S,x)} \max_{e \in S} \left\{ \frac{u_e - x_e}{u_e - l_e} \right\}, \text{ subject to}$$

$$\sum_{e \in S} x_e \leq B,$$

$$S \in \mathcal{S}(E),$$

$$x_e \in [l_e, u_e],\ e \in S.$$

The problem Min-Max-Risk is introduced by Hu (2010) and it is motivated by routing and network design applications. In the routing application, x_e is the travel time, and $r_e(x_e)$ is the travel risk over the segment e of a network. The objective is to find a simple path between two specified nodes of the network and to determine the travel time over each segment of this path such that the maximum segment risk is minimized and the total travel time is upper bounded by a given number B. In the network design application, x_e is the cost of establishing a connection between two nodes of the edge e of an undirected network, and $r_e(x_e)$ is the risk of failure of this connection. The objective is to find a spanning tree comprising the established connections and to determine the connection cost for each edge of this tree such that the maximum edge failure risk is minimized and the total cost is upper bounded by B. Hu (2010) develops $O(n^5)$ and $O(nm^3 \log m)$ algorithms for the "simple path" and "spanning tree" variants of the Min-Max-Risk problem, respectively, by employing the general idea of the *parametric search* of Megiddo (1979).

Several problems similar to Min-Max-Risk have been studied in the literature. Particularly, Álvarez-Miranda et al. (2014) (Section 3.1) investigate a min-max problem defined on Steiner trees of a series-parallel graph, and Chen et al. (2009a, b) and Álvarez-Miranda et al. (2011, 2014) explore min-sum problems defined on simple paths, spanning trees and Steiner trees.

Scheduling problems with job processing parameters controllable via allocation of a limited non-renewable resource have been also studied, among others, by Cheng et al. (1998, 2006), Grigoriev et al. (2007), Janiak and Kovalyov (1996), Janiak et al. (2005), Ng et al. (2003), Shabtay and Steiner (2007), and Shioura et al. (2015).

In the next section, we establish new properties of the problem Min-Max-Risk and cite those presented by Hu (2010). Then we employ these properties to reduce the problem Min-Max-Risk, in which $\mathcal{S}(E)$ is the set of simple paths or spanning trees, to a problem with a rational objective function, and contrary to Hu (2010), employ Megiddo (1979) parametric search directly, which drastically reduces the solution time of Min-Max-Risk in these cases. Section 3 presents a different solution approach based on the bisection search. The paper concludes with a short summary of the results and suggestions for extensions and future research.

2 Problem properties and direct parametric search

In this section we make some useful simplifying assumptions, cite properties established by Hu (2010), make new observations about the problem MIN-MAX-RISK, and directly employ Megiddo (1979) parametric search to solve this problem for the "simple path" and "spanning tree" cases.

2.1 Simplifying assumptions and known properties

Observe that the problem MIN-MAX-RISK has a feasible solution if and only if there exists $S \in \mathcal{S}(E)$ such that $\sum_{e \in S} l_e \leq B$, or equivalently, $\min_{S \in \mathcal{S}(E)} \sum_{e \in S} l_e \leq B$. If $\mathcal{S}(E)$ is the set of simple paths, then the latter condition is verified by solving the shortest path problem with the non-negative arc weights l_e, $e \in E$. This can be done, for example, by employing the original algorithm of Dijkstra (1959) or the theoretically best $O(m + n \log n)$ algorithm of Fredman and Tarjan (1987). If $\mathcal{S}(E)$ is the set of spanning trees, then the condition is verified by solving the minimum weight spanning tree problem with the non-negative edge weights l_e, $e \in E$. This can be done, for example, by the original algorithm of Borůvka (1926) (with a tie-breaking mechanism for common edge weights) or the theoretically best $O(m\alpha(m, n))$ algorithm of Chazelle (2000), where $\alpha(\cdot, \cdot)$ is the inverse Ackermann function.

Let r^* denote the optimal solution value of the problem MIN-MAX-RISK. Obviously, $0 \leq r^* \leq 1$. Furthermore, $r^* = 0$ if and only if there exists $S \in \mathcal{S}(E)$ such that $\sum_{e \in S} u_e \leq B$, or equivalently, $\min_{S \in \mathcal{S}(E)} \sum_{e \in S} u_e \leq B$. If $\mathcal{S}(E)$ is the set of simple paths (or spanning trees), then the latter condition is verified by solving the shortest path problem (resp. the minimum spanning tree problem) with the arc (resp. edge) weights u_e, $e \in E$. We exclude the trivial case $r^* = 0$ from further consideration, and without loss of generality, accept the following assumption.

Assumption 1 The problem MIN-MAX-RISK has a solution with optimal value r^* such that $0 < r^* \leq 1$, or equivalently, (1) there exists $S \in \mathcal{S}(E)$ satisfying $\sum_{e \in S} l_e \leq B$, and (2) $\sum_{e \in S} u_e > B$ for all $S \in \mathcal{S}(E)$.

The following two properties are established by Hu (2010).

Property 1 (Hu 2010) If the problem MIN-MAX-RISK has a solution, then there exists an optimal solution (S^*, x^*) of this problem such that $r^* = \frac{u_e - x_e^*}{u_e - l_e}$ for each element $e \in S^*$.

It follows from Property 1 that, if the problem MIN-MAX-RISK has a solution, then the optimal values x_e^* can be calculated as $x_e^* = u_e - r^*(u_e - l_e)$, $e \in S^*$.

Property 2 (Hu 2010) An optimal solution (S^*, x^*) of the problem MIN-MAX-RISK satisfies the equality $\sum_{e \in S^*} x_e^* = B$.

2.2 New observations and direct application of Megiddo's parametric search

Let us introduce the function $H(S) = \frac{\sum_{e \in S} u_e - B}{\sum_{e \in S}(u_e - l_e)}$ and the following auxiliary problem.

$$\textbf{Problem } \textsc{Min-H}: \min_{S \in \mathcal{S}(E)} H(S).$$

In view of Assumption 1, $H(S) > 0$ for any $S \in \mathcal{S}(E)$. Furthermore, if the problem MIN-MAX-RISK has a solution, then, taking into account Properties 1 and 2, we obtain $B = \sum_{e \in S^*} x_e^* = \sum_{e \in S^*} u_e - r^* \sum_{e \in S^*}(u_e - l_e)$, which implies $r^* = \frac{\sum_{e \in S^*} u_e - B}{\sum_{e \in S^*}(u_e - l_e)}$. In other words, r^* is an upper bound on the optimal solution value of the problem MIN-H . The following property is crucial for the application of the parametric search approach to solve the problem MIN-MAX-RISK.

Property 3 The optimal solution value of the problem MIN-H is equal to r^*.

Proof Let r^0 denote the optimal solution value of the problem MIN-H . Then $r^0 \leq H(S^*) = r^* \leq 1$. We will show that $r^0 = r^*$. Suppose the contrary that there exists a structure $S^0 \in \mathcal{S}(E)$ with value $H(S^0) = r^0 < r^*$. Then, by setting $x_e^0 = u_e - r^0(u_e - l_e)$, $e \in S^0$, we obtain $x_e^0 \in [l_e, u_e]$ and $r_e(x_e^0) = \frac{u_e - x_e^0}{u_e - l_e} = r^0$ for any $e \in S^0$. Moreover, since $r^0 = H(S^0) = \frac{\sum_{e \in S^0} u_e - B}{\sum_{e \in S^0}(u_e - l_e)}$, we deduce $\sum_{e \in S^0} x_e^0 = \sum_{e \in S^0} u_e - r^0 \sum_{e \in S^0}(u_e - l_e) = B$. Therefore, (S^0, x^0) is a feasible solution of the problem MIN-MAX-RISK, whose maximum risk value r^0 is less than r^*, which contradicts the optimality of (S^*, x^*). $\square$

Property 3 implies that the problem MIN-MAX-RISK reduces to the problem MIN-H, which is an example of the optimization problem with a rational objective function studied by Megiddo (1979) and Radzik (1998).

Consider 0–1 variables y_i, $i = 1, \ldots, k$. Denote $y = (y_1, \ldots, y_k)$. Let D be a feasible domain of the variable vectors y. The following two problems are studied in Megiddo (1979).

$$\textbf{Problem } \text{P1}: \min_{y \in D} \sum_{i=1}^{k} c_i y_i, \qquad \textbf{Problem } \text{P2}: \min_{y \in D} \frac{a_0 + \sum_{i=1}^{k} a_i y_i}{b_0 + \sum_{i=1}^{k} b_i y_i}.$$

It is assumed that the denominator in the problem P2 is always positive. Let t^* denote the optimal solution value of the problem P2. Suppose that lower and upper bounds $t_{\min}$ and $t_{\max}$ are known such that $t_{\min} \leq t^* \leq t_{\max}$. The following theorem is proved by Megiddo.

Theorem 1 (Megiddo 1979) *If there exists a "basic" algorithm which solves the problem* P1 *within $O(T)$ elementary operations for any fixed weights $c_i = a_i - tb_i$, $i = 1, \ldots, k$, where $t_{\min} \leq t \leq t_{\max}$, then the problem* P2 *can be solved in time $O(T^2)$.*

The problem MIN-H is a special case of the problem P2, in which $k = |E| = m$, $a_0 = -B$, $b_0 = 0$, $a_i = u_i$, $b_i = u_i - l_i$, $i \in E$, $D = \{y = (y_1, \ldots, y_m) \mid \{i : y_i = 1\} \in \mathcal{S}(E)\}$, $t_{\min} = 0$ and $t_{\max} = 1$. Observe that the weights $c_i = u_i - t(u_i - l_i)$ are non-negative for any t satisfying $t_{\min} \leq t \leq t_{\max}$. Therefore, the $O(m + n \log n)$ time algorithm of Fredman and Tarjan (1987) for the shortest path problem and the $O(m\alpha(m, n))$ time algorithm of Chazelle (2000) for the minimum weight spanning tree problem can serve as the basic algorithm mentioned in Theorem 1.

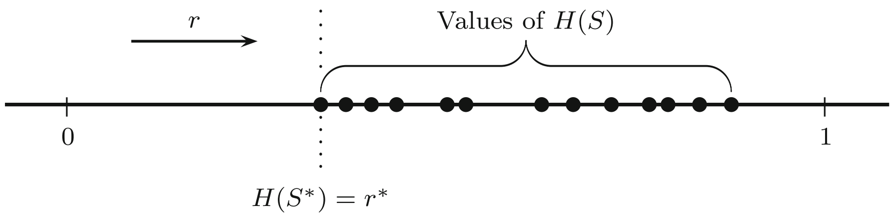

Fig. 1 Visual interpretation of the maximization problem

It follows from Theorem 1 that the problem MIN-H , and hence, the problem MIN-MAX-RISK is solvable in $O(m^2 + n^2(\log n)^2)$ and $O(m^2(\alpha(m, n))^2)$ time for the cases of simple paths and spanning trees, respectively. For the case of spanning trees, Megiddo (1979) presents an alternative algorithm for the problem P2 with $O(T \log n)$ running time, where $O(T)$ is the running time of the basic algorithm, which reduces the solution time of MIN-H to $O(m\alpha(m, n) \log n)$ in this case. Radzik (1998) presents an $O(mn)$ time algorithm for the maximization counterpart of the problem P2, in which D is the set of paths between two specified nodes of an *acyclic* graph. It can be used for the problem MIN-MAX-RISK in this case.

3 Bisection search

We have shown in the previous section that the problem MIN-MAX-RISK is equivalent to the problem MIN-H , which, in its turn, is equivalent to the following maximization problem, whose visual interpretation is given in Fig. 1.

$$\max_{r\in[0,1]} r, \text{ subject to}$$

$$r \leq H(S), \ \forall S \in \mathcal{S}(E). \tag{1}$$

The optimal solution value of the problem MIN-MAX-RISK is equal to the maximum value of r, for which $r \leq \min\{H(S) \mid S \in \mathcal{S}(E)\}$. Observe that, if the relation (1) is satisfied for a certain r, then it is satisfied for all $r' < r$, and if it is not satisfied for a certain r, then it is not satisfied for all $r' > r$. This observation can be employed to solve the problem MIN-MAX-RISK by a bisection search over the range [0, 1] of the min-max risk values r. In each iteration of the bisection search, the relation (1) is verified. It is easy to see that (1) can be re-written as

$$B \leq \sum_{e\in S}(u_e - r(u_e - l_e)), \ \forall S \in \mathcal{S}(E),$$

which is equivalent to

$$B \leq \min_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - r(u_e - l_e)).$$

Table 1 Asymptotic running times

Problem	Hu (2010)	Parametric search for MIN-H	Bisection search for MIN-H
Simple path	$O(n^5)$	$O(m^2+n^2(\log n)^2)$	$O((m+n\log n)\log W)$
Spanning tree	$O(nm^3\log m)$	$O(m\alpha(m,n)\log n)$	$O(m\alpha(m,n)\log W)$

The verification of (1) for a given r, $0 \le r \le 1$, reduces to solving the shortest path problem or the minimum weight spanning tree problem with fixed non-negative arc (resp. edge) weights $u_e - r(u_e - l_e)$.

In order to be finite and efficient, the bisection search requires a sufficient gap between the optimal min-max risk value $r^* = H(S^*)$ and any other min-max risk value $r = H(S)$ in the problem MIN-H . Since $H(S) = \frac{\sum_{e\in S} u_e - B}{\sum_{e\in S}(u_e - l_e)}$, this gap cannot be less than $1/W^2$, where $W \ge \max_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - l_e)$, for example, $W = \sum_{e\in E}(u_e - l_e)$. Below we give a formal description of the bisection search. It operates with iteratively adjusted lower and upper bounds L and U such that $L \le r^* \le U$. Solution of the problem $\min_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - r(u_e - l_e))$ is denoted as $S^{(r)}$.

Bisection search.

Step 1. Set $L = 0$ and $U = 1$. Determine $S^{(L)}$.

Step 2. If $|U - L| \le 1/W^2$, then output structure $S^* = S^{(L)}$ with risk value $r^* = H(S^*)$ and stop. If $|U - L| > 1/W^2$, then compute $r = (L + U)/2$. If $B \le \min_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - r(u_e - l_e))$, then re-set $L := r$ and repeat Step 2. If $B > \min_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - r(u_e - l_e))$, then re-set $U := r$ and repeat Step 2.

The bisection search can be implemented to run in $O(T \log W)$ time, where T is the time of solving the problem $\min_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - r(u_e - l_e))$. Recall that if $\mathcal{S}(E)$ is the set of simple paths, then $T = O(m + n\log n)$, and if $\mathcal{S}(E)$ is the set of spanning trees, then $T = O(m\alpha(m, n))$. Table 1 presents the asymptotic running times of the algorithms in Hu (2010) and the two solution methods in this paper.

4 Conclusions, extensions and suggestions for future research

In this paper, two new solution approaches for the problem MIN-MAX-RISK are suggested, which are faster than those suggested earlier in Hu (2010). Note that the reverse problem

$$\min_{(S,x)} \sum_{e\in S} x_e, \text{ subject to}$$

$$\max_{e\in S}\left\{\frac{u_e-x_e}{u_e-l_e}\right\} \le A,$$

$$S \in \mathcal{S}(E),$$

$$x_e \in [l_e, u_e],\ e \in S,$$

is equivalent to

$$\min_{(S,x)} \sum_{e\in S} x_e, \text{ subject to}$$

$$S \in \mathcal{S}(E),$$

$$x_e \in [\max\{l_e, u_e - A(u_e - l_e)\}, u_e],\ e \in S,$$

and, since $0 \le A \le 1$, it finally reduces to $\min_{S\in\mathcal{S}(E)} \sum_{e\in S}(u_e - A(u_e - l_e))$. Therefore, if $\mathcal{S}(E)$ is the set of simple paths or the set of spanning trees, then the reverse problem is as easy as the corresponding shortest path problem or the minimum weight spanning tree problem.

The bi-criteria problem with two independent criteria to find solutions (S, x), which minimize the maximum risk $F_1 = \max_{e\in S}\left\{\frac{u_e-x_e}{u_e-l_e}\right\}$ and the total budget $F_2 = \sum_{e\in S} x_e$, can be of a practical interest. An approximation of the set of *Pareto optimal* solutions for this problem can be constructed by employing the general *ε-constraint* method, described, for example, in Cheng et al. (1998), Ehrgott (2005) or T'kindt and Billaut (2006). Let δ_1 and δ_2 be given absolute deviations of an approximate solution from a Pareto optimal solution with respect to the maximum risk and the total budget, respectively. Assume that $0 < \delta_1, \delta_2 \le 1$. There can be two variants of the ε-constraint method.

In the first variant, the problem MIN-MAX-RISK is solved for $B = \delta_2, 2\delta_2, \ldots, U_2$, where U_2 is an upper bound on the total budget for the Pareto optimal solutions, which is a multiple of δ_2, for example, $U_2 = \left\lceil \frac{\sum_{e\in E} u_e}{\delta_2} \right\rceil \delta_2$. The set of approximate solutions can be constructed in $O(T_2 U_2/\delta_2)$ time, where T_2 is the time of solving the problem MIN-MAX-RISK. It is easy to see that, for any Pareto optimal solution with values (F_1, F_2) of the maximum risk and the total budget, there exists an approximate solution with the corresponding pair of values (F_1', F_2') such that $F_1' \le F_1$ and $F_2' \le F_2 + \delta_2$.

In the second variant, the inverse problem is solved for $A = \delta_1, 2\delta_1, \ldots, U_1$, where $U_1 = \min\{1, \left\lceil \frac{1}{\delta_1} \right\rceil \delta_1\}$. Again, it is easy to see that, for any Pareto optimal solution with values (F_1, F_2) of the maximum risk and the total budget, there exists an approximate solution with the corresponding values (F_1', F_2') such that $F_1' \le F_1 + \delta_1$ and $F_2' \le F_2$. The set of approximate solutions for the second variant can be constructed in $O(T_1 \left\lceil \frac{1}{\delta_1} \right\rceil)$ time, where T_1 is the time of solving the reverse problem.

In the future, it makes sense to study problems in which variables x_e are integer and to consider other practically interesting examples of the sets E and $\mathcal{S}(E)$. The studied controllable risk model can be useful in other application areas such as robust scheduling (Sevaux and Sörensen 2004), stock options portfolio margining (Coffman et al. 2010), portfolio assets allocation (Scutellà and Recchia 2010) and manufacturer's pricing decisions under the risk of demand uncertainty (Arcelus et al. 2011).

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

Álvarez-Miranda E, Chen X, Hu J, Hu X, Candia-Véjar A (2011) Deterministic risk control for cost-effective network connections. Theor Comput Sci 412(3):257–264

Álvarez-Miranda E, Candia-Véjar A, Chen XJ, Hu XD, Li B (2014) Risk models for the prize collecting steiner tree problems with interval data. Acta Math Appl Sin Engl Ser 30(1):1–26

Arcelus FJ, Kumar S, Srinivasan G (2011) Channel coordination with manufacturer's return policies within a newsvendor framework. 4OR 9(3):279–297

Borůvka O (1926) O jistém problému minimálním. Práce Moravské Přírodovědecké Společnosti 3(3):37–53

Chazelle B (2000) A minimum spanning tree algorithm with inverse-Ackermann type complexity. J ACM 47(6):1028–1047

Chen X, Hu J, Hu X (2009a) A new model for path planning with interval data. Comput Oper Res 36(6):1893–1899

Chen X, Hu J, Hu X (2009b) A polynomial solvable minimum risk spanning tree problem with interval data. Eur J Oper Res 198(1):43–46

Cheng TCE, Janiak A, Kovalyov MY (1998) Bicriterion single machine scheduling with resource dependent processing times. SIAM J Optim 8(2):617–630

Cheng TCE, Kovalyov MY, Shakhlevich NV (2006) Scheduling with controllable release dates and processing times: Makespan minimization. Eur J Oper Res 175(2):751–768

Coffman EG Jr, Matsypura D, Timkovsky VG (2010) Strategy vs risk in margining portfolios of options. 4OR 8(4):375–386

Dijkstra EW (1959) A note on two problems in connexion with graphs. Numer Math 1(1):269–271

Ehrgott M (2005) Multicriteria optimization, 2nd edn. Springer, Berlin

Fredman ML, Tarjan RE (1987) Fibonacci heaps and their uses in improved network optimization algorithms. J ACM 34(3):596–615

Grigoriev A, Sviridenko M, Uetz M (2007) Machine scheduling with resource dependent processing times. Math Program Ser B 110(1):209–228

Hu J (2010) Minimizing maximum risk for fair network connection with interval data. Acta Math Appl Sin Engl Ser 26(1):33–40

Janiak A, Kovalyov MY (1996) Single machine scheduling subject to deadlines and resource dependent processing times. Eur J Oper Res 94(2):284–291

Janiak A, Kovalyov MY, Kubiak W, Werner F (2005) Positive half-products and scheduling with controllable processing times. Eur J Oper Res 165(2):416–422

Megiddo N (1979) Combinatorial optimization with rational objective functions. Math Oper Res 4(4):414–424

Ng CT, Cheng TCE, Kovalyov MY (2003) Batch scheduling with controllable setup and processing times to minimize total completion time. J Oper Res Soc 54(5):499–506

Radzik T (1998) Fractional combinatorial optimization. In: Du DZ, Pardalos PM (eds) Handbook of combinatorial optimization, vol 3. Springer, New York, pp 429–478

Scutellà MG, Recchia R (2010) Robust portfolio asset allocation and risk measures. 4OR 8(2):113–139

Sevaux M, Sörensen K (2004) A genetic algorithm for robust schedules in a one-machine environment with ready times and due dates. 4OR 2(2):129–147

Shabtay D, Steiner G (2007) A survey of scheduling with controllable processing times. Discrete Appl Math 155:1643–1666

Shioura A, Shakhlevich NV, Strusevich VA (2015) Decomposition algorithms for submodular optimization with applications to parallel machine scheduling with controllable processing times. Math Program Ser A 153(2):495–534

T'kindt V, Billaut JC (2006) Multicriteria scheduling: theory, models and algorithms, 2nd edn. Springer, Berlin