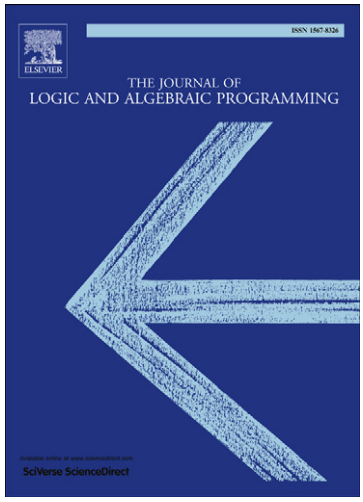


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# Consistency and Refinement for Interval Markov Chains[1,2]

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**Abstract**

Interval Markov Chains (IMC), or Markov Chains with probability intervals in the transition matrix, are the base of a classic specification theory for probabilistic systems (Larsen and Jonsson, 1991). The standard semantics of IMCs assigns to a specification the set of all Markov Chains that satisfy its interval constraints. The theory then provides operators for deciding emptiness of conjunction and refinement (entailment) for such specifications.

In this paper we study complexity of several problems for IMCs, that stem from compositional modeling methodologies. In particular, we close the complexity gap for thorough refinement of two IMCs and for deciding the existence of a common implementation for an unbounded number of IMCs, showing that these problems are EXPTIME-complete.

We discuss suitable notions of determinism for specifications, and show that for deterministic IMCs the syntactic refinement operators are complete with respect to model inclusion. Finally, we show that deciding consistency (emptiness) for an IMC is polynomial and that existance of common implementation can be established in polynomial time for any constant number of IMCs.

*Keywords:* Markov Chain, Abstraction, Refinement, Complexity, Determinism

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## 1. Introduction

Interval Markov Chains (IMCs for short) extend Markov Chains, by allowing to specify intervals of possible probabilities on state transitions. IMCs have been introduced by Larsen and Jonsson [1] as a *specification* formalism—a basis for a stepwise-refinement-like modeling method, where initial designs are very abstract and underspecified, and then they are made continuously more precise, until they are concrete. Unlike richer specification models, such as Constraint Markov Chains [2], IMCs are difficult to use for compositional specification due to lack of basic modeling operators. To address this, we study complexity and algorithms for deciding consistency of conjunctive sets of IMC specifications.

Let us consider an example. Figure 1 presents a simple specification of a user of coffee machine. The model on the left hand side prescribes that a typical user orders coffee with milk with probability $x \in [0, 0.5]$ and black coffee with probability $y \in [0.2, 0.7]$ (customers also buy tea with probability $t \in [0, 0.5]$).

Jonsson and Larsen [1] have introduced refinement of such processes, but have not characterized its computational complexity. Refinement allows deciding whether one specification allows a subset of the probabilistic processes allowed by another one. We extend the work on refinement by classifying its complexity and characterizing it using structural coinductive algorithms in the style of simulation.

Consider the issue of combining multiple specifications of the same system. It turns out that conjunction of IMCs cannot be expressed as an IMC itself, due to a lack of expressiveness of intervals. We have recently shown this formally in a parallel work [3]. Here we illustrate this with an example. The right hand side model in Figure 1 presents a different view on the coffee service. The vendor of the machine delivers another specification, which prescribes that the machine is serviceable only if coffee (white or black) is ordered with some probability $z \in [0.4, 0.8]$ from among other beverages, otherwise it will run out of coffee powder too frequently, or the powder becomes too old. A conjunction of these two

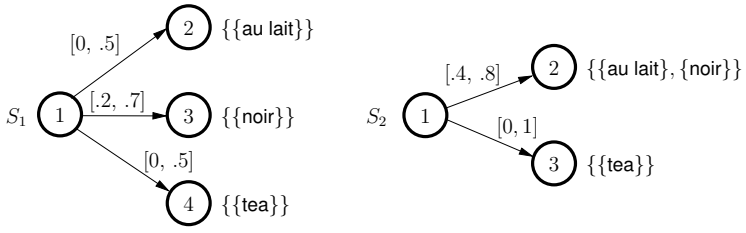



Figure 1: Two specifications of different aspects of a coffee service

models would describe usage patterns compatible with this particular machine. Such a conjunction effectively requires that all the interval constraints are satisfied and that $z = x + y$ holds. However, the solution of this constraint is not described by an interval over $x$ and $y$. This can be seen by pointing out an extremal point, which is not a solution, while all its coordinates take part in some solution. Say $x = 0$ and $y = 0.2$ violates the interval for $z$, while for each of these two values it is possible to select another one in such a way that $z$'s constraint is also held (for example $(x = 0, y = 0.4)$ and $(x = 0.2, y = 0.2)$). Thus the solution space is not an interval over $x$ and $y$. This lack of closure properties for IMCs motivates us to address the problem of reasoning about conjunction, without constructing it — the, so called, common implementation problem.

In this paper we provide algorithms and complexities for thorough refinement, consistency, common implementation, and refinement of IMCs, in order to enable compositional modeling. We contribute the following new results:

- We define suitable notions of determinism for IMCs, and show that for deterministic IMCs *thorough refinement* (TR) coincides with two simulation-like preorders (the *weak refinement* and *strong refinement*), for which there exist co-inductive algorithms terminating in a polynomial number of iterations.

- In [1] a TR between IMCs is defined as an inclusion of their implementation sets. We show that the procedure for deciding TR given in [1] can be implemented in single exponential time. Furthermore, we provide a lower bound, concluding that TR is EXPTIME-complete. While the reduction from TR of modal transition systems [4] used to provide this lower bound is conceptually simple, it requires a rather involved proof of correctness, namely that it preserves sets of implementations in a sound and complete manner.

- A polynomial procedure for checking whether an IMC is *consistent* (C), i.e. it admits an implementation as a Markov Chain.

- An exponential procedure for checking whether $k$ IMCs are consistent in the sense that they share a Markov Chain satisfying all—a *common implementation* (CI). We show that this problem is EXPTIME-complete.

- As a special case, we observe that CI is PTIME for any constant value of $k$. In particular, checking whether two specifications can be simultaneously satisfied, and synthesizing their shared implementation can be done in polynomial time.

The paper proceeds as follows. We begin by summarizing prior work on these and related problems, and surveying application areas for Interval Markov Chains (Section 2). In Section 3 we introduce the basic definitions. All results in subsequent sections are new and ours. In Section 4 we discuss deciding TR and other refinement procedures. We expand on the interplay of determinism and refinements in Section 5. The problems of C and CI are addressed in Section 6. We conclude by discussing the results in Section 7.

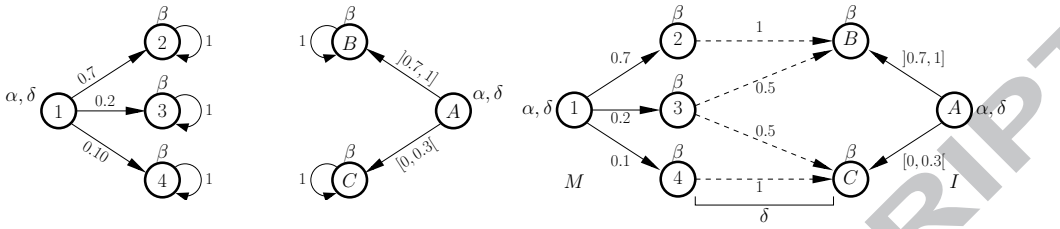
## 2. State of The Art

Besides IMCs, there exists many other specification formalisms for describing and analyzing stochastic systems; the list includes process algebras [5, 6] or logical frameworks [7]. A logical representation is suited for conjunction. The process algebraic specifications tend to be well developed for parallel composition and efficient refinement checking. For example, it is not clear how one can synthesize a MC (an implementation) that satisfies two Probabilistic Computation Tree Logic formulas. Similarly, conjunction is usually not defined for process algebraic specifications. In this sense, IMCs situate themselves in the middle between logical and process algebraic models—one can reason about their common implementation and refinement.

In mathematics, the abstraction of Markov set-chains [8] lies very close to IMCs. The latter defines the intervals on the transition probabilities, while the former uses matrix intervals in the transition matrix space, which allows reasoning about the abstraction using linear algebra. Technically, a Markov set-chain is an explicit enumeration of all the implementations of an IMC. Markov set-chains have been, for instance, used to approximate dynamics of hybrid systems [9]. Arguably, they have a different objective and compositional reasoning operators have not been considered for them, so far.

IMCs have served the purpose of *abstraction* in model checking, where a concrete system is being soundly abstracted by a less precise system in order to prove the properties more easily [10, 11, 12, 13]. The main issues related to model checking of IMCs have recently been addressed in [12].

As we already stated, IMCs are not expressive enough to represent many artifacts of compositional design. In [2], we have presented Constraint Markov Chains (CMC) a specification model that, contrary to IMCs, is closed under composition and conjunction. While more expressive than IMCs, CMCs are not an immediate and universal replacement for IMCs, given that complexity of decision procedures for them is much higher. IMCs remain relevant, whenever parallel




(a) A Markov Chain $M$ (b) An IMC $I$ (c) An example of satisfaction relation

Figure 2: Markov Chain, Interval Markov Chain and satisfaction relation

composition is not required in the application, or when they are used as a coarse abstraction (for example) for CMCs.

For functional analysis of discrete-time non-probabilistic systems, the theory of Modal Transition Systems (MTS) [14, 15] provides a specification formalism supporting refinement, conjunction and parallel composition. Earlier we have obtained EXPTIME-completeness both for the corresponding notion of CI [16] and of TR [4] for MTSs. In [1] it is shown that IMCs properly contain MTSs, which puts our new results in a somewhat surprising light: in the complexity theoretic sense, and as far as CI and TR are considered, the generalization of modalities by probabilities does come for free. A recent overview of research on (discrete) modal specifications is available in [17].

## 3. Background

We shall now introduce the basic definitions used throughout the paper. In the following we will write $\mathsf{Intervals}_{[0,1]}$ for the set of all closed, half-open and open intervals included in $[0, 1]$.

A Markov Chain (sometimes MC in short) is a tuple $C = \langle P, p_0, \pi, A, V_C \rangle$, where $P$ is a set of states containing the initial state $p_0$, $A$ is a set of atomic propositions, $V_C : P \to 2^A$ is a state valuation labeling states with propositions, and $\pi : P \to \mathsf{Distr}(P)$ is a probability distribution assignment such that $\sum_{p' \in P} \pi(p)(p') = 1$ for all $p \in P$. The probability distribution assignment is the only component that is relaxed in IMCs:

**Definition 1 (Interval Markov Chain).** *An* Interval Markov Chain *is a tuple* $I = \langle Q, q_0, \varphi, A, V_I \rangle$*, where* $Q$ *is a finite set of states containing the initial state* $q_0$*,* $A$ *is a set of atomic propositions,* $V_I : Q \to 2^A$ *is a state valuation, and* $\varphi : Q \to (Q \to \mathsf{Intervals}_{[0,1]})$*, which for each* $q \in Q$ *and* $q' \in Q$ *gives an interval of probabilities.*

Instead of a distribution, as in MCs, in IMCs we have a function mapping elementary events (target states) to intervals of probabilities. We interpret this function as a constraint over distributions. This is expressed in our notation as follows. Given a state $q \in Q$ and a distribution $\sigma \in \mathsf{Distr}(Q)$, we say that $\sigma \in \varphi(q)$ iff $\sigma(q') \in \varphi(q)(q')$ for all $q' \in Q$. Occasionally, it is convenient to think of a Markov Chain as an IMC, in which all probability intervals are closed point intervals.

We visualize IMCs as automata with intervals on transitions. As an example, consider the IMC in Figure 2b. It has two outgoing transitions from the initial state $A$. No arc is drawn between states if the probability is zero (or, more precisely, the interval is $[0, 0]$), so in the example there is zero probability of going from state $A$ to $A$, or from $B$ to $C$, etc. Otherwise, the probability distribution over successors of $A$ is constrained to fall into $]0.7, 1]$ and $[0, 0.3]$ for $B$ and $C$ respectively. States $B$ and $C$ have valuation $\beta$, whereas state $A$ has valuation $\alpha, \delta$. Please observe that Figure 2a presents a Markov Chain using the same convention, modulo the intervals. Remark that our formalism does not allow "sink states", i.e. states with no outgoing transition. However, in order to avoid clutter in the figures, we sometimes represent states with no outgoing transitions. They must be interpreted as states with a self-loop with a closed point interval consisting of probability 1.

A *satisfaction* relation establishes compatibility of Markov Chains (implementations) and IMCs (specifications). The original definition of satisfaction between MCs and IMCs was presented in [1, 18]. We use a slightly modified, but strictly equivalent definition using a concept of *correspondence functions*:

**Definition 2 (Satisfaction).** *Let $C = \langle P, p_0, \pi, A, V_C\rangle$ be a MC and let $I = \langle Q, q_0, \varphi, A, V_I\rangle$ be an IMC. A relation $\mathcal{R} \subseteq P \times Q$ is called a satisfaction relation if whenever $p\,\mathcal{R}\,q$ then*

- *Their valuation sets agree: $V_C(p) = V_I(q)$*
- *There exists a correspondence function $\delta : P \to (Q \to [0, 1])$ such that*
  1. *For all $p' \in P$, if $\pi(p)(p') > 0$ then $\delta(p')$ defines a distribution on $Q$,*
  2. *$\sum_{p' \in P} \pi(p)(p')\delta(p')(q') \in \varphi(q)(q')$ for all $q' \in Q$, and*
  3. *if $\delta(p')(q') > 0$, then $p'\,\mathcal{R}\,q'$.*

We write $C \models I$ iff there exists a satisfaction relation containing $(p_0, q_0)$. $C$ is an *implementation* of $I$. The set of implementations of $I$ is written $[\![I]\!]$. Figure 2c presents an example of satisfaction on states 1 and $A$. The correspondence function is specified using labels on the dashed arrows i.e. the probability mass going from state 1 to 3 is distributed to state $B$ and $C$ with half going to each.

We will say that a state $q$ of an IMC is *consistent* if its interval constraint $\varphi(q)$ is satisfiable, i.e., there exists a distribution $\sigma \in \mathsf{Distr}(Q)$ satisfying $\varphi(q)$. Obviously, for a given IMC, it is sufficient that all its states are consistent in order to guarantee that the IMC is consistent itself—there exists a Markov Chain satisfying it. We discuss the problem of establishing consistency in a sound and complete manner in Section 6.

There are three known ways of defining refinement for IMCs: the strong refinement (introduced as *simulation* in [1]), weak refinement (introduced under the name of *probabilistic simulation* in [12]), and thorough refinement (introduced as *refinement* in [1]). We will recall their formal definitions:

**Definition 3 (Strong Refinement).** *Let $I_1 = \langle Q, q_0, \varphi_1, A, V_1\rangle$ and $I_2 = \langle S, s_0, \varphi_2, A, V_2\rangle$ be two IMCs. A relation $\mathcal{R} \subseteq Q \times S$ is called a* strong refinement relation *if whenever $q\,\mathcal{R}\,s$, then*

- *Their valuation sets agree: $V_1(q) = V_2(s)$ and*
- *There exists a correspondence function $\delta : Q \to (S \to [0,1])$ such that for all $\sigma \in \mathsf{Distr}(Q)$, if $\sigma \in \varphi_1(q)$, then*
  1. *for each $q' \in Q$ such that $\sigma(q') > 0$, $\delta(q')$ is a distribution on $S$,*
  2. *for all $s' \in S$, we have $\sum_{q' \in Q} \sigma(q')\delta(q')(s') \in \varphi_2(s)(s')$, and*
  3. *for all $q' \in Q$ and $s' \in S$, if $\delta(q')(s') > 0$, then $q'\,\mathcal{R}\,s'$.*

*$I_1$ strongly refines $I_2$, written $I_1 \leq_{\mathsf{S}} I_2$, iff there exists a strong refinement relation containing $(q_0, s_0)$.*

A strong refinement relation requires existence of a single correspondence, which witnesses satisfaction for any resolution of probability constraint over successors of $q$ and $s$. Figure 3a illustrates such a correspondence between states $A$ and $\alpha$ of two IMCs. The correspondence function is given by labels on the dashed lines. It is easy to see that regardless of how the probability constraints are resolved the correspondence function distributes the probability mass in a fashion satisfying $\alpha$.

We now recall the notion of *weak refinement*, first introduced in [12] under the name of probabilistic simulation.

**Definition 4 (Weak Refinement).** *Let $I_1 = \langle Q, q_0, \varphi_1, A, V_1\rangle$ and $I_2 = \langle S, s_0, \varphi_2, A, V_2\rangle$ be two IMCs. A relation $\mathcal{R} \subseteq Q \times S$ is called a* weak refinement relation *if whenever $q\,\mathcal{R}\,s$, then*

- *Their valuation sets agree:* $V_1(q) = V_2(s)$
- *For each* $\sigma \in \mathsf{Distr}(Q)$ *such that* $\sigma \in \varphi_1(q)$*, there exists a correspondence function* $\delta : Q \to (S \to [0,1])$ *such that*
  1. *For each* $q' \in Q$ *such that* $\sigma(q') > 0$*,* $\delta(q')$ *is a distribution on* $S$*,*
  2. *for all* $s' \in S$*, we have* $\sum_{q' \in Q} \sigma(q')\delta(q')(s') \in \varphi_2(s)(s')$*, and*
  3. *for all* $q' \in Q$ *and* $s' \in S$*, if* $\delta(q')(s') > 0$*, then* $q' \mathcal{R} s'$*.*

*$I_1$ weakly refines $I_2$, written $I_1 \leq_{\mathsf{W}} I_2$, iff there exists a weak refinement relation containing $(q_0, s_0)$.*

The weak refinement between two states requires that, for any resolution of probability constraint over successors in $I_1$, there exists a correspondence function which witnesses satisfaction of $I_2$. Thus the weak refinement achieves the weakening by swapping the order of quantifications. Figure 3b illustrates such a correspondence between states $A$ and $\alpha$ of another two IMCs. Here, $x$ stands for a value in $[0.2, 1]$ (arbitrary choice of probability of going to state $C$ from $A$). Notably, for each choice of $x$, there exists $p \in [0, 1]$ such that $px \in [0, 0.6]$ and $(1 - p)x \in [0.2, 0.4]$. Remark that strong refinement naturally implies weak refinement. Indeed, if there exists a single correspondence function witnessing satisfaction for any resolution of the constraints, then there exists a correspondence function for each resolution of the constraints.

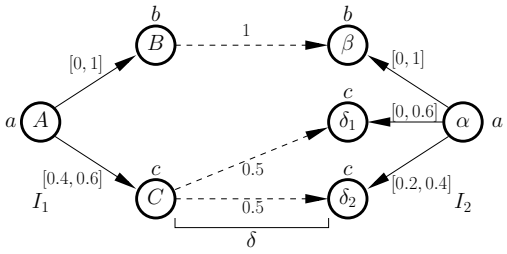
Finally, we introduce the thorough refinement as defined in [1]:

**Definition 5 (Thorough Refinement).** *IMC $I_1$* thoroughly refines *IMC $I_2$, written $I_1 \leq_{\mathsf{T}} I_2$, iff each implementation of $I_1$ implements $I_2$: $[\![I_1]\!] \subseteq [\![I_2]\!]$*

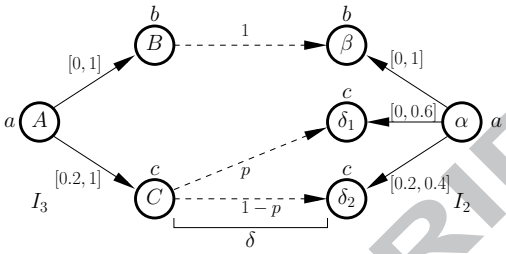
Thorough refinement is the ultimate refinement relation for any specification formalism, as it is based on the semantics of the models.

## 4. Refinement Relations

We will now compare the expressiveness of the refinement relations. It is not hard to see that both strong and weak refinements soundly approximate the thorough refinement (since they are transitive and degrade to satisfaction if the left argument is a Markov Chain). The converse does not hold. We will now discuss procedures to compute weak and strong refinements, and then compare the granularity of these relations, which will lead us to procedures for computing thorough




(a) Illustration of a strong refinement relation between an IMC $I_1$ and an IMC $I_2$




(b) A weak refinement relation between an IMC $I_3$ and an IMC $I_2$; $p$ is a parameter

Figure 3: Illustration of strong and weak refinement relations

refinement. Observe that all three refinement are decidable as they only rely on the first order theory of real numbers. In concrete cases below the calculations can be done more efficiently due to convexity of solution spaces for interval constraints.

*Weak and Strong Refinement.* Consider two IMCs $I_1 = \langle P, o_1, \varphi_1, A, V_1 \rangle$ and $I_2 = \langle Q, o_2, \varphi_2, A, V_2 \rangle$. Informally, checking whether a given relation $\mathcal{R} \subseteq P \times Q$ is a weak refinement relation reduces to checking, for each pair $(p, q) \in \mathcal{R}$, whether the following formula is true: $\forall \pi \in \varphi_1(p)\ \exists \delta : P \to (Q \to [0, 1])$ such that $\pi\delta$ satisfies a system of linear equations / inequalities. Since the set of distributions satisfying $\varphi_1(p)$ is convex, checking such a system is exponential in the number of variables, here $|P||Q|$. As a consequence, checking whether a relation on $P \times Q$ is a weak refinement relation is exponential in $|P||Q|$. For strong refinement relations, the only difference appears in the formula that must be checked: $\exists \delta : P \to (Q \to [0, 1])$ such that $\forall \pi \in \varphi_1(p)$, we have that $\pi\delta$ satisfies a system of linear equations / inequalities. Therefore, checking whether a relation on $P \times Q$ is a strong refinement relation is also exponential in $|P||Q|$.

Deciding whether weak (strong) refinement holds between $I_1$ and $I_2$ can be done in the usual coinductive fashion by considering the total relation $P \times Q$ and successively removing all the pairs that do not satisfy the above formulae. The refinement holds iff the relation we reach contains the pair $(o_1, o_2)$. The algorithm will terminate after at most $|P||Q|$ iterations. This gives an upper bound on the complexity to establish strong and weak refinements: a polynomial number of iterations over an exponential step. This upper bound may be loose. One could try to reuse techniques for non-stochastic systems [19] in order to reduce the number of iterations. This is left to future work.

*Granularity.* In [1] an informal statement is made that the strong refinement is strictly stronger (finer) than the thorough refinement: $(\leq_{\mathsf{T}}) \supsetneq (\leq_{\mathsf{S}})$. In [12], the weak refinement is introduced without discussing its relations to neither the strong nor the thorough refinement. The following theorem resolves all open issues in relations between the three:

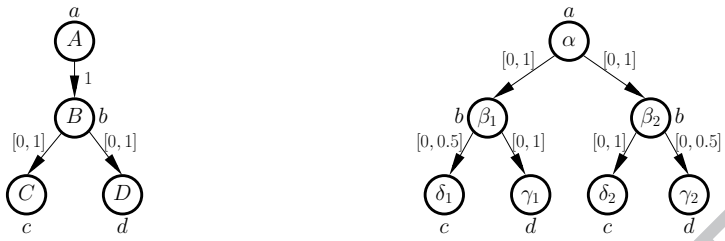
**Theorem 1.** *The thorough refinement is strictly weaker than the weak refinement, which is strictly weaker than the strong refinement:* $(\leq_{\mathsf{T}}) \supsetneq (\leq_{\mathsf{W}}) \supsetneq (\leq_{\mathsf{S}})$.

*Proof.*

First, remark that weak refinement implies thorough refinement. Indeed, weak refinement is transitive and degrades to satisfaction when its left argument is a Markov chain. Thus it is equivalent to say that a MC $M$ satisfies an IMC $I$ and that $M \preceq_W I$. If furthermore $I \preceq_W I'$, then, by transitivity, we obtain $M \preceq_W I'$, which is equivalent to $M \models I'$. As a consequence, if $I \preceq_W I'$, then for all MC $M$ such that $M \models I$, it holds that $M \models I'$, i.e. $[\![I]\!] \subseteq [\![I']\!]$.

We now consider the two inequalities separately.

1. Case 1: $(\leq_{\mathsf{T}}) \supsetneq (\leq_{\mathsf{W}})$. Figure 4 proposes two IMCs $I_4$ and $I_5$, such that $I_4$ thoroughly but not weakly refines $I_5$. Indeed, let $M = \langle Q, q_0, \pi, \{a, b, c, d\}, V_M \rangle$ be an implementation of $I_4$ and $\mathcal{R}$ a corresponding satisfaction relation. Let $P \subseteq Q$ be the set of states of $M$ satisfying $B$. Consider a state $p \in P$. Let $\pi^C(p) = \sum_{\{q \in Q \mid q \,\mathcal{R}\, C\}} \pi(p)(q)$ and $\pi^D(p) = \sum_{\{q \in Q \mid q \,\mathcal{R}\, D\}} \pi(p)(q)$. Since $p \,\mathcal{R}\, B$, we have that $\pi^C(p) + \pi^D(p) = 1$. Let $P_1 \subset P$ be the set of states of $M$ such that $\pi^C(p) \leq 0.5$ and let $P_2 \subset P$ be the set of states of $M$ such that $\pi^D(p) < 0.5$. Obviously, we have $P = P_1 \cup P_2$ and $P_1 \cap P_2 = \emptyset$. By construction, the states in $P_1$ will satisfy $\beta_1$ and the states in $P_2$ will satisfy $\beta_2$. We now build a satisfaction relation $\mathcal{R}'$ such that, for all $q \in M$, if $q \,\mathcal{R}\, A$, then $q \,\mathcal{R}\, \alpha$ ; if $q \in P_1$, then $q \,\mathcal{R}'\, \beta_1$ ; if $q \in P_2$, then $q \,\mathcal{R}'\, \beta_2$ ; if $q \,\mathcal{R}\, C$, then $q \,\mathcal{R}'\, \delta_1$ and $q \,\mathcal{R}'\, \delta_2$ ; and if $q \,\mathcal{R}\, D$ then $q \,\mathcal{R}'\, \gamma_1$ and $q \,\mathcal{R}'\, \gamma_2$. By construction, $\mathcal{R}'$ is a satisfaction relation, and $M$ is an implementation of $I_5$. Thus, $[\![I_4]\!] \subseteq [\![I_5]\!]$. However, it is not possible to define a weak refinement relation between $I_4$ and $I_5$: obviously, $B$ can neither refine $\beta_1$ nor $\beta_2$.
2. Case 2: $(\leq_{\mathsf{W}}) \supsetneq (\leq_{\mathsf{S}})$. In Figure 3b, we propose two IMCs, $I_3$ and $I_2$ such that $I_3$ weakly but not strongly refines $I_2$. State $A$ weakly refines state $\alpha$: Given a value $x$ for the transition $A \to C$, we can split it in order to match both transitions $\alpha \xrightarrow{px} \delta_1$ and $\alpha \xrightarrow{(1-p)x} \delta_2$. Define $\delta(C)(\delta_1) = p$ and




(a) IMC $I_4$ (b) IMC $I_5$

Figure 4: IMCs $I_4$ and $I_5$ such that $I_4$ thoroughly but not weakly refines $I_5$

$\delta(C)(\delta_2) = (1-p)$, with

$$p = \begin{cases} 0 & \text{if } 0.2 \leq x \leq 0.4 \\ \frac{x-0.3}{x} & \text{if } 0.4 < x < 0.8 \\ 0.6 & \text{if } 0.8 \leq x \end{cases}$$

$\delta_1$ is a correspondence function witnessing a weak refinement relation between $A$ and $\alpha$. Consider the following parametric inequalities, where $p$ is the variable and $x$ the parameter.

$$\begin{aligned} xp &\leq 0.6 \\ x(1-p) &\leq 0.4 \\ x(1-p) &\geq 0.2 \end{aligned} \tag{1}$$

Suppose that a strong refinement relation $\mathcal{R}$ exists between $I_3$ and $I_2$. Then the correspondence function witnessing $A \,\mathcal{R}\, \alpha$ should be similar to the one given above, where $p$ would be a constant solution of the system of inequalities (1). However, one can see from the solutions of this system of inequalities, which are graphically represented in Figure 5, that there exists no value of $p$ satisfying (1) for all $x$.

□

*Deciding Thorough Refinement.* As weak and strong refinements are strictly stronger than thorough refinement, it is interesting to investigate the complexity of deciding TR. In [1] a procedure computing TR is given, albeit without a complexity class. We now establish the complexity of this procedure, closing the problem:

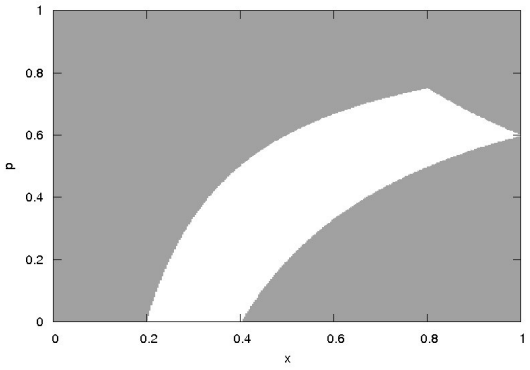



Figure 5: Solutions (in white) of the system of inequalities (1)

**Theorem 2.** *The decision problem TR of establishing whether there exists a thorough refinement between two given IMCs is EXPTIME-complete.*

The proofs for both the upper and the lower bounds rely on a series of results that are presented in the rest of this section.

**The upper bound.** The upper-bound is shown by analyzing the complexity of the algorithm presented in [1]. For the sake of completeness, and in order to clarify several typesetting inaccuracies of the original presentation, we quote the construction of [1] below and subsequently analyze its complexity:

**Definition 6 (Subset simulation).** *Let* $I_1 = \langle Q, q_0, \varphi_Q, A, V_Q \rangle$ *and* $I_2 = \langle P, p_0, \varphi_P, A, V_P \rangle$ *be IMCs. A total relation* $\mathcal{R} \subseteq Q \times 2^P$ *is a* subset-simulation *iff for each state* $q \in Q$*:*

1. $q \,\mathcal{R}\, T$ *implies* $V_Q(q) = V_P(t)$ *for all* $t \in T$
2. *For each probability distribution* $\pi_Q \in \varphi_Q(q)$ *and each correspondence function* $\delta_Q : Q \to (2^P \to [0,1])$ *such that* $\mathsf{support}(\delta_Q) \subseteq \mathcal{R}$*, there exists a set* $T$ *such that* $q \,\mathcal{R}\, T$ *and for each* $t \in T$*, there exists a probability distribution* $\pi_P \in \varphi_P(t)$ *and a correspondence function* $\delta_P : P \to (2^P \to [0,1])$ *such that*
   (a) *if* $\delta_P(t')(T') > 0$*, then* $t' \in T'$*, and*
   (b) *for all* $T' \in 2^P$*, we have*

$$\sum_{q' \in Q} \pi_Q(q')\delta_Q(q')(T') = \sum_{p' \in P} \pi_P(p')\delta_P(p')(T').$$

Intuitively, this relation associates to every state $q$ of $I_1$ a sample of sets of states $(T_1, \ldots, T_k)$ of $I_2$ that are "compatible" with $q$. Then, for each admissible redistribution $\delta$ of the successor states of $q$, it states that there exists one of the sets $T_i$ such that for each of its states $t'$, there is a redistribution $\gamma$ of the successor states of $t'$ that is compatible with $\delta$. In [1] it is shown that the existence of a subset-simulation between two IMCs $I_1$ and $I_2$ is equivalent to thorough refinement between them. We now propose an example to illustrate the subset simulation algorithm presented above.

**Example 1.** *Consider the IMCs* $I_4 = \langle \{A, B, C, D\}, A, \varphi_4, \{a, b, c, d\}, V_4 \rangle$ *and* $I_5 = \langle \{\alpha, \beta_1, \beta_2, \delta_1, \delta_2, \gamma_1, \gamma_2\}, \alpha, \varphi_5, \{a, b, c, d\}, V_5 \rangle$ *given in Figure 4. They are such that* $I_4$ *thoroughly but not weakly refines* $I_5$ *(c.f. proof of Theorem 1). Since thorough refinement holds, we can exhibit a subset simulation* $\mathcal{R} \subseteq P \times 2^Q$ *between* $I_4$ *and* $I_5$*: Let* $\mathcal{R} = \{(A, \{\alpha\}), (B, \{\beta_1\}), (B, \{\beta_2\}), (C, \{\delta_1, \delta_2\}), (D, \{\gamma_1, \gamma_2\})\}$ *be this subset simulation. We illustrate the unfolding of* $\mathcal{R}$ *for states* $A$ *and* $B$ *of* $I_4$*. The rest is left to the reader.*

*Consider state* $A$ *of* $I_4$*.*

1. *We have* $A \mathcal{R} \{\alpha\}$*, and* $V_4(A) = a = V_5(\alpha)$*.*
2. *The only distribution* $\pi \in \varphi_4(A)$ *is such that* $\pi(B) = 1$*. Let for example* $\Delta^1 \in [0, 1]^{4 \times 2^7}$ *be the correspondence matrix such that* $\Delta^1_{B,\{\beta_1\}} = 1/2$ *and* $\Delta^1_{B,\{\beta_2\}} = 1/2$*. Let* $\{\alpha\}$ *be the set such that* $A \mathcal{R} \{\alpha\}$*. Let* $\rho$ *be the distribution on* $Q$ *such that* $\rho(\beta_1) = \rho(\beta_2) = 1/2$*.* $\rho$ *is indeed in* $\varphi_5(\alpha)$*. Let* $\Delta^2 \in [0, 1]^{7 \times 2^7}$ *be the correspondance matrix such that* $\Delta^2_{\beta_1,\{\beta_1\}} = 1$ *and* $\Delta^2_{\beta_2,\{\beta_2\}} = 1$*. It is then obvious that*
   (a) *for all* $t$ *and* $T$*, if* $\Delta^2_{t,T} > 0$*, then* $t \in T$*;*
   (b) $\pi \Delta^1 = \rho \Delta^2$ *holds.*

*Consider state* $B$ *of* $I_4$*.*

1. *We have* $B \mathcal{R} \{\beta_1\}$ *and* $B \mathcal{R} \{\beta_2\}$*. It holds that* $V_4(B) = b = V_5(\beta_1) = V_5(\beta_2)$*.*
2. *Consider a distribution* $\pi \in \varphi_4(B)$ *(for example such that* $\pi(C) < 1/2$*). Let* $\Delta^1$ *be an admissible correspondance matrix. We must have* $\Delta^1_{C,\{\delta_1,\delta_2\}} = 1$ *and* $\Delta^1_{D,\{\gamma_1,\gamma_2\}} = 1$*. Consider* $\{\beta_1\}$ *the set such that* $B \mathcal{R} \{\beta_1\}$ *(if* $\pi(C) > 1/2$ *then pick up* $\{\beta_2\}$ *instead). Let* $\rho$ *be the distribution such that* $\rho(\delta_1) = \pi(C)$ *and* $\rho(\gamma_1) = \pi(D)$*. Since* $\pi(C) < 1/2$*, we have* $\rho \in \varphi_5(\beta_1)$*. Let* $\Delta^2$ *be a correspondance matrix such that* $\Delta^2_{\delta_1,\{\delta_1,\delta_2\}} = 1$ *and* $\Delta^2_{\gamma_1,\{\gamma_1,\gamma_2\}} = 1$*. It is obvious that*

(a) *for all* $t$ *and* $T$*, if* $\Delta^2_{t,T} > 0$*, then* $t \in T$*;*
(b) $\pi\Delta^1 = \rho\Delta^2$ *holds.*

*The rest of the unfolding is obvious, and* $\mathcal{R}$ *is thus a subset simulation.*

The existence of a subset simulation between two IMCs is decided using a standard co-inductive fixpoint calculation. The algorithm works as follows: first consider the total relation and check whether it is a subset-simulation. Then refine it by removing violating pairs of states, and check again until a fixpoint is reached (it becomes a subset-simulation or it is empty). Checking whether a given relation is a subset simulation has a single exponential complexity. Checking the second condition in the definition can be done in single exponential time by solving polynomial constraints with fixed quantifiers for each pair $(q, T)$ in the relation. There are at most $|Q|2^{|P|}$ such pairs, which gives a single exponential time bound for the cost of one iteration of the fixpoint loop. There are at most $|Q|2^{|P|}$ elements in the total relation and at least one is removed in an iteration, which gives $O(|Q|2^{|P|})$ as the bound on the number of iterations. Since a polynomial of two exponentials is still an exponential, we obtain a single exponential time for running time of this computation.

**Remark 1.** *Summarizing, all three refinements are in EXPTIME. Still, weak refinement seems easier to check than thorough. For TR the number of iterations on the state-space of the relation is exponential while it is only polynomial for the weak refinement. Also, the constraint solved at each iteration involves a single quantifier alternation for the weak, and three alternations for the thorough refinement.*

**The Lower Bound.** The lower bound of Theorem 2 is shown by a polynomial reduction of the thorough refinement problem for modal transition systems to TR of IMCs. The former problem is known to be EXPTIME-complete [4].

A modal transition system (an MTS in short) [15] is a tuple $M = (S, s_0, A, \rightarrow, \dashrightarrow)$, where $S$ is the set of states, $s_0$ is the initial state, and $\rightarrow \subseteq S \times A \times S$ are the transitions that *must* be taken and $\dashrightarrow \subseteq S \times A \times S$ are the transitions that *may* be taken. In addition, it is assumed that $(\rightarrow) \subseteq (\dashrightarrow)$.

A modal transition system $M = (S, s_0, A, \rightarrow, \dashrightarrow)$ refines another modal transition system $N = (T, t_0, A, \rightarrow, \dashrightarrow)$ iff there exists a refinement relation $R \subseteq S \times T$ containing $(s_0, t_0)$ such that if $(s, t) \in R$, then

1. whenever $t \xrightarrow{a} t'$ then also $s \xrightarrow{a} s'$ for some $s' \in S$ and $(s', t') \in R$

2. whenever $s \overset{a}{\dashrightarrow} s'$ then also $t \overset{a}{\dashrightarrow} t'$ for some $t' \in T$ and $(s', t') \in R$

A labelled transition system *implements* a MTS if it refines it in the above sense. Thorough refinement of MTSs is defined as inclusion of implementation sets, analogously to IMCs.

We now describe a translation of MTSs into IMCs which preserves implementations. We assume we only work with modal transition systems that have no deadlock-states, in the sense that each state has at least one outgoing must transition. This assumption is needed to avoid dealing with inconsistent states in the corresponding IMC. We first present a transformation that takes any two MTS and transforms them into MTS without deadlocks, preserving the notion of thorough refinement between them.

Let $M = \langle S, s_0, A, \rightarrow, \dashrightarrow \rangle$ be a MTS. Let $\perp \notin A$ be a new action variable, and $q \notin S$ be a new state variable. Define a new MTS $M_\perp = \langle S \cup \{q\}, s_0, A \cup \{\perp\}, \rightarrow_\perp, \dashrightarrow_\perp \rangle$ as follows: for all $s, s' \in S$ and $a \in A$, $s \overset{a}{\rightarrow}_\perp s' \iff s \overset{a}{\rightarrow} s'$ and $s \overset{a}{\dashrightarrow}_\perp s' \iff s \overset{a}{\dashrightarrow} s'$. Add the following transitions: for all $s \in S \cup \{q\}$, $s \overset{\perp}{\rightarrow}_\perp q$ and $s \overset{\perp}{\dashrightarrow}_\perp q$. In this way, every state of $M_\perp$ has at least one must outgoing transition. Moreover, it is trivial to see that this transformation preserves the notion of thorough refinement. This is stated in the following theorem:

**Theorem 3.** *Let $M$ and $M'$ be two MTS. If $\perp$ is in neither of their sets of actions, we have $[\![M]\!] \subseteq [\![M']\!] \iff [\![M_\perp]\!] \subseteq [\![M'_\perp]\!]$.*

Finally, we can safely suppose that all the MTS we consider in the rest of the section have no deadlocks.

We now describe an implementation preserving translation of MTSs into IMCs. The IMC $\widehat{M}$ corresponding to a MTS $M$ is defined by the tuple $\widehat{M} = \langle Q, q_0, A \cup \{\epsilon\}, \varphi, V \rangle$ where $Q = S \times (\{\epsilon\} \cup A)$, $q_0 = (s_0, \epsilon)$, for all $(s, x) \in Q$, $V((s, x)) = \{x\}$ and $\varphi$ is defined as follows : for all $t, s \in S$ and $b, a \in (\{\epsilon\} \cup A)$, $\varphi((t, b))((s', a)) = ]0, 1]$ if $t \overset{a}{\rightarrow} s$ ; $\varphi((t, b))((s', a)) = [0, 0]$ if $t \not\overset{a}{\dashrightarrow} s$ ; and $\varphi((t, b))((s', a)) = [0, 1]$ otherwise. The encoding is illustrated in Figure 6.

We first state two lemmas that will be needed to prove the main theorem of the section: the encoding presented above reduces the problem of checking thorough refinement on modal transition systems to checking thorough refinement on IMCs.

**Lemma 4.** *Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$ be an MTS and $I = (S_I, s_0^I, A, \rightarrow)$ be a transition system. We have $I \models M \Rightarrow [\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!]$.*


(a) A MTS $M$ (b) The IMC $\widehat{M}$

Figure 6: An example of the translation from Modal Transition Systems to IMCs

*Proof.* We first recall the definition of a satisfaction relation for MTS: Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$ be an MTS and $I = (S_I, s_0^I, A, \rightarrow)$ be a transition system. The implementation $I$ satisfies the MTS $M$, written $I \models M$, iff there exists a relation $\mathcal{R} \subseteq S_I \times S$ such that

1. $s_0^I \,\mathcal{R}\, s_0$
2. Whenever $s_I \,\mathcal{R}\, s$, we have
   (a) For all $a \in A$, $s_I' \in S_I$, $s_I \xrightarrow{a} s_I'$ in $I$ implies that there exists $s' \in S$ such that $s \overset{a}{\dashrightarrow} s'$ in $M$ and $s_I' \,\mathcal{R}\, s'$.
   (b) For all $a \in A$, $s' \in S$, $s \xrightarrow{a} s'$ in $M$ implies that there exists $s_I' \in S_I$ such that $s_I \xrightarrow{a} s_I'$ in $M$ and $s_I' \,\mathcal{R}\, s'$.

Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$ be an MTS and $I = (S_I, s_0^I, A, \rightarrow)$ be a transition system. Let $\widehat{M} = \langle Q, q_0, A \cup \{\epsilon\}, \varphi, V\rangle$ and $\widehat{I} = \langle Q_I, (s_0^I, \epsilon), A \cup \{\epsilon\}, \varphi_I, V_I\rangle$ be the IMCs defined as above.
Suppose that $I \models M$. By definition, there exists a satisfaction relation for MTS $\mathcal{R} \subseteq S_I \times S$ such that $s_0^I \,\mathcal{R}\, s_0$. We show that $[\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!]$.

Let $T = \langle Q_T, p_0, \pi^T, V_T, A\rangle$ be an MC such that $T \in [\![\widehat{I}]\!]$. By definition, there exists a satisfaction relation for IMCs $\mathcal{R}_1 \subseteq Q_T \times Q_I$ such that $p_0 \,\mathcal{R}_1 (s_0^I, \epsilon)$. Define the new relation $\mathcal{R}_2 \subseteq Q_T \times Q$ such that $p \,\mathcal{R}_2 (s, x)$ iff there exists $s_I \in S_I$ such that $p \,\mathcal{R}_1 (s_I, x)$ and $s_I \,\mathcal{R}\, s$. We show that $\mathcal{R}_2$ is a satisfaction relation between $T$ and $\widehat{M}$.

Let $p, s, s_I, x$ be such that $p \,\mathcal{R}_1 (s_I, x)$ and $s_I \,\mathcal{R}\, s$, i.e. $p \,\mathcal{R}_2 (s, x)$. If $x \neq \bot$, we have

1. Since $p \,\mathcal{R}_1 (s_I, x)$, we have $V_T(p) = V_I((s_I, x)) = \{x\}$. Thus $V_T(p) = V((s, x)) = \{x\}$.

2. Let $\delta^1 \in \mathsf{Distr}(Q_T \times Q_I)$ be the probability distribution witnessing $p\,\mathcal{R}_1(s_I, x)$, and let $\delta^2 \in \mathsf{Distr}(Q_T \times Q)$ be the correspondence matrix such that for all $p' \in Q_T,\ s' \in S$ and $y \in A$, if $\{s'_I \in S_I \mid s'_I\,\mathcal{R}\,s'\} \neq \emptyset$ and $s \overset{y}{\dashrightarrow} s'$, then

$$\delta^2(p', (s', y)) = \sum_{\{s'_I \in S_I \mid s'_I\,\mathcal{R}\,s'\}} \frac{\delta^1(p', (s'_I, y))}{|\{s'' \in S \mid s'_I\,\mathcal{R}\,s'' \text{ and } s \overset{y}{\dashrightarrow} s''\}|};$$

Otherwise, $\delta^2(p', (s', y)) = 0$.
Recap that we suppose that all must transitions are also may transitions. The definition above potentially gives a non-zero value to $\delta^2(p', (s', y))$ if there exists a may (or must) transition from $s$ to $s'$ in $S$ labelled with $y$ and if there exists a state $s'_I$ in $I$ such that $s'_I\,\mathcal{R}\,s'$.
Let $p' \in Q_T$. We prove that $\sum_{(s',y)} \delta_2(p', (s', y)) = \pi^T(p)(p')$: By definition of $\delta^1$, we have $\sum_{(s'_I,y)} \delta^1(p', (s'_I, y)) = \pi^T(p)(p')$.

$$\sum_{(s',y)} \delta^2(p', (s', y)) =$$

$$\sum_{\{(s',y) \mid \exists s'_I,\ s'_I\,\mathcal{R}\,s' \text{ and } s \overset{y}{\dashrightarrow} s'\}} \ \sum_{\{s'_I \mid s'_I\,\mathcal{R}\,s'\}} \frac{\delta^1(p', (s'_I, y))}{|\{s'' \in S \mid s'_I\,\mathcal{R}\,s'' \text{ and } s \overset{y}{\dashrightarrow} s''\}|}.$$

Clearly, for all $(s'_I, y)$ such that $\delta^1(p', (s'_I, y)) > 0$, the term $\frac{\delta^1(p', (s'_I, y))}{|\{s'' \in S \mid s'_I\,\mathcal{R}\,s'' \text{ and } s \overset{y}{\dashrightarrow} s''\}|}$ will appear exactly $|\{s'' \in S \mid s'_I\,\mathcal{R}\,s'' \text{ and } s \overset{y}{\dashrightarrow} s''\}|$ times in the expression above. As a consequence, $\sum_{(s',y)} \delta^2(p', (s', y)) = \sum_{(s'_I,y)} \delta^1(p', (s'_I, y)) = \pi^T(p)(p')$.
Moreover, we show that for all $(s', y) \in Q$, that $\sum_{p' \in Q^T} \delta^2(p', (s', y)) \in \varphi((s, x)(s', y))$. By construction, $\varphi((s, x)(s', y))$ is either $\{0\}$, $[0, 1]$ or $]0, 1]$. We will thus prove that (a) if $\sum_{p' \in Q^T} \delta^2(p', (s', y)) > 0$, then $\varphi((s, x)(s', y)) \neq \{0\}$; and (b) if $\varphi((s, x)(s', y)) = ]0, 1]$, then $\sum_{p' \in Q^T} \delta^2(p', (s', y)) > 0$.

(a) Suppose $\sum_{p' \in Q^T} \delta^2(p', (s', y)) > 0$. By definition, there must exist $p'$ such that $\delta^2(p', (s', y)) > 0$. As a consequence, by definition of $\delta^2$, there exists a transition $s \overset{y}{\dashrightarrow} s'$ in $M$ and $\varphi((s, x), (s', y)) \neq \{0\}$.

(b) If $\varphi((s, x)(s', y)) = ]0, 1]$, then there exists a transition $s \overset{y}{\to} s'$ in $M$. As a consequence, by $\mathcal{R}$, there exists $s'_I \in S_I$ such that $s_I \overset{y}{\to} s'_I$

in $I$ and $s'_I \mathcal{R} s'$. Thus $\varphi_I((s_I, x), (s'_I, y)) = ]0, 1]$. By definition of $\delta^1$, we know that $\sum_{p' \in Q^T} \delta^1(p', (s'_I, y)) > 0$, thus there exists $p' \in Q_T$ such that $\delta^1(p', (s'_I, y)) > 0$. Since $s'_I \mathcal{R} s'$ and $s \xrightarrow{y} s'$, we have $\delta^2(p', (s', y)) > 0$, thus $\sum_{p'' \in Q^T} \delta^2(p'', (s', y)) > 0$.

Finally, if $\delta^2(p', (s', y)) > 0$, there exists $s'_I \in S_I$ such that $s'_I \mathcal{R} s'$ and $\delta^1(p', (s'_I, y)) > 0$. By definition of $\delta^1$, we have $p' \mathcal{R}_1(s'_I, y)$. As a consequence, $p' \mathcal{R}_2(s', y)$.

$\mathcal{R}_2$ satisfies the axioms of a satisfaction relation for IMCs, thus $T \in [\![\widehat{M}]\!]$ and finally $[\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!]$.

□

**Lemma 5.** *Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$be an MTS and $I = (S_I, s_0^I, A, \rightarrow)$ be a transition system. We have $[\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!] \Rightarrow I \models M$.*

*Proof.*

Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$be an MTS and $I = (S_I, s_0^I, A, \rightarrow)$ be a transition system. Let $\widehat{M} = \langle Q, q_0, A \cup \{\epsilon\}, \varphi, V \rangle$ and $\widehat{I} = \langle Q_I, q_0^I, A \cup \{\epsilon\}, \varphi_I, V_I \rangle$ be the IMCs defined as above.

Suppose that $[\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!]$. We prove that $I \models M$.

Let $T = \langle Q_T, p_0, \pi^T, V_T, A \rangle$ be an MC such that $T \in [\![\widehat{I}]\!]$. As a consequence, there exists two satisfaction relations for IMCs $\mathcal{R}_1 \subseteq Q_T \times Q_I$ and $\mathcal{R}_2 \subseteq Q_T \times Q$ such that $p_0 \mathcal{R}_1(s_0^I, \epsilon)$ and $p_0 \mathcal{R}_2(s_0, \epsilon)$. Define the new relation $\mathcal{R} \subseteq S_I \times S$ such that $s_I \mathcal{R} s$ iff there exists $p \in Q_T$ and $x \in (\{\epsilon\} \cup A)$ such that $p \mathcal{R}_1(s_I, x)$ and $p \mathcal{R}_2(s, x)$. We have

1. $p_0 \mathcal{R}_1(s_0^I, \epsilon)$ and $p_0 \mathcal{R}_2(s_0, \epsilon)$. As a consequence, $s_0^I \mathcal{R} s_0$.
2. Let $s_I, s, p, x$ such that $p \mathcal{R}_1(s_I, x)$ and $p \mathcal{R}_2(s, x)$ and let $\delta^1 \in \mathsf{Distr}(Q_T \times Q_I)$ and $\delta^2 \in \mathsf{Distr}(Q_T \times Q)$ be the associated probability distributions.
   (a) Let $y \in A$ and $s'_I \in S_I$ such that $s_I \xrightarrow{y} s'_I$ in $I$. We prove that there exists $s' \in S$ such that $s \xdashrightarrow{y} s'$ and $s'_I \mathcal{R} s'$.
   By definition of $\widehat{I}$, we have $\varphi_I((s_I, x), (s'_I, y)) = ]0, 1]$. As a consequence, $\sum_{p'' \in Q_T} \delta^1(p'', (s'_I, y)) > 0$. Thus there exists $p'$ in $Q_T$ such that $\delta^1(p', (s'_I, y)) > 0$. By definition of $\delta^1$, we have $p' \mathcal{R}_1(s'_I, y)$, thus $V_T(p') = V_I((s'_I, y)) = \{y\}$.
   Moreover, by definition of $\delta^1$, we have $\sum_{(s''_I, z) \in Q_I} \delta^1(p', (s''_I, z)) = \pi^T(p)(p')$. Since $\delta^1(p', (s'_I, y)) > 0$, we have $\pi^T(p)(p') > 0$.

By definition of $\delta^2$, we know that $\sum_{(s'',z)\in Q} \delta^2(p',(s'',z)) = \pi^T(p)(p') > 0$. As a consequence, there exists $(s',z) \in Q$ such that $\delta^2(p',(s',z)) > 0$. By definition of $\delta^2$,we have $p' \mathcal{R}_2(s',z)$ and since $V_T(p') = \{y\}$, we must have $z = y$.
Consequently, $\sum_{p''\in Q_T} \delta^2(p'',(s',y)) > 0$. By definition of $\delta^2$, we know that
$\sum_{p''\in Q_T} \delta^2(p'',(s',y)) \in \varphi((s,x),(s',y))$, thus $\varphi((s,x),(s',y)) \neq \{0\}$, which means, by definition of $\widehat{M}$, that there exists a transition $s \overset{y}{\dashrightarrow} s'$ in $M$. Moreover, there exits $p' \in Q_T$ such that both $p' \mathcal{R}_1(s'_I,y)$ and $p' \mathcal{R}_2(s',y)$, thus $s'_I \mathcal{R} s'$.

(b) Let $y \in A$ and $s' \in S$ such that $s \overset{y}{\rightarrow} s'$ in $M$. We prove that there exists $s'_I \in S_I$ such that $s_I \overset{y}{\rightarrow} s'_I$ in $I$ and $s'_I \mathcal{R} s'$.
By definition of $\widehat{M}$, we have $\varphi((s,x),(s',y)) = ]0,1]$. As a consequence,
$\sum_{p''\in Q_T} \delta^2(p'',(s',y)) > 0$. Thus there exists $p'$ in $Q_T$ such that $\delta^2(p',(s',y)) > 0$. By definition of $\delta^2$, we have $p' \mathcal{R}_2(s',y)$, thus $V_T(p') = V((s',y)) = \{y\}$.
Moreover, by definition of $\delta^2$, we have $\sum_{(s'',z)\in Q} \delta^2(p',(s'',z)) = \pi^T(p)(p')$. Since $\delta^2(p',(s',y)) > 0$, we have $\pi^T(p)(p') > 0$.
By definition of $\delta^1$, we know that $\sum_{(s''_I,z)\in Q_I} \delta^1(p',(s''_I,z)) = \pi^T(p)(p') > 0$. As a consequence, there exists $(s'_I,z) \in Q_I$ such that $\delta^1(p',(s'_I,z)) > 0$. By definition of $\delta^1$, we have $p' \mathcal{R}_1(s'_I,z)$ and since $V_T(p') = \{y\}$, we must have $z = y$.
Consequently, $\sum_{p''\in Q_T} \delta^1(p'',(s'_I,y)) > 0$. By definition of $\delta^1$,we know that $\sum_{p''\in Q_T} \delta^1(p'',(s'_I,y)) \in \varphi_I((s_I,x),(s'_I,y))$, thus $\varphi_I((s,x),(s',y)) \neq \{0\}$, which means, by definition of $\widehat{I}$, that there exists a transition $s_I \overset{y}{\rightarrow} s'_I$ in $I$ (remember that $I$ is a classical transition system). Moreover, there exits $p' \in Q_T$ such that both $p' \mathcal{R}_1(s'_I,y)$ and $p' \mathcal{R}_2(s',y)$, thus $s'_I \mathcal{R} s'$.
Finally, $\mathcal{R}$ is a satisfaction relation for MTS, and $I \models M$

□

From the two lemmas stated above, we can infer the following theorem:

**Theorem 6.** *Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$be an MTS and $I = (S_I, s_0^I, A, \rightarrow)$ be a transition system. We have $I \models M \iff [\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!]$.*

We now define a construction $f$ that builds, for all implementations $C$ of $\widehat{M}$, a corresponding implementation $f(C)$ of $M$:

Let $M = (S, s_0, A, \rightarrow, \dashrightarrow)$ be a MTS. Let $\widehat{M} = \langle S \times (\{\epsilon\} \cup A), (s_0, \epsilon), \{\epsilon\} \cup A, \varphi, V \rangle$ be the transformation of $M$ defined as above. Let $C = \langle Q, q_0, A, \pi, V' \rangle$ be a MC such that $C \models \widehat{M}$ for some satisfaction relation on IMCs $\mathcal{R}$. Define $f(C) = (Q, q_0, A, \rightarrow)$ the Transition System such that $q \xrightarrow{a} q'$ whenever $\pi(q, q') > 0$ and $V'(q') = \{a\}$. By construction, it is trivial that (1) $f(C) \models M$ for some satisfaction relation on MTS $\mathcal{R}'$ and (2) $C \models \widehat{f(C)}$ for some satisfaction relation on IMCs $\mathcal{R}''$. These satisfaction relations are defined as follows:

- $q \,\mathcal{R}'\, s$ whenever there exists $x \in \{\epsilon\} \cup A$ such that $q \,\mathcal{R}(s, x)$ ;
- $q \,\mathcal{R}''(q', x)$ whenever $q = q'$.

We now switch to the main theorem, showing that the transformation $M \rightarrow \widehat{M}$ indeed preserves thorough refinement.

**Theorem 7.** *Let $M$ and $M'$ be two Modal Transition Systems and $\widehat{M}$ and $\widehat{M'}$ be the corresponding IMCs defined as above. We have $M \leq_{\mathsf{T}} M' \iff \widehat{M} \leq_{\mathsf{T}} \widehat{M'}$.*

*Proof.* Let $M$ and $M'$ be two MTS, and $\widehat{M}$ and $\widehat{M'}$ the corresponding IMCs.

- $\Rightarrow$ Suppose that $M \preceq_T M'$, and let $C$ be a MC such that $C \models \widehat{M}$. We have by construction $f(C) \models M$, thus $f(C) \models M'$. By Theorem 6, we have $[\![\widehat{f(C)}]\!] \subseteq [\![\widehat{M'}]\!]$, and we know that $C \models \widehat{f(C)}$. As a consequence, $C \models \widehat{M'}$.
- $\Leftarrow$ Suppose that $\widehat{M} \preceq_T \widehat{M'}$, and let $I$ be a TS such that $I \models M$. By Theorem 6, we have $[\![\widehat{I}]\!] \subseteq [\![\widehat{M}]\!]$, thus by hypothesis $[\![\widehat{I}]\!] \subseteq [\![\widehat{M'}]\!]$. Finaly, by Theorem 6, we obtain that $I \models M'$.

□

Crucially, this translation is polynomial. Thus if we had a subexponential algorithm for TR of IMCs, we could use it to obtain a subexponential algorithm for TR of MTSs, which is impossible [4].

## 5. Determinism

Humans naturally build deterministic models to represent deterministic implementations. Thus deterministic objects form an important class of specifications. It is also known that for other specification langages, determinism allows more efficient reasoning procedures.

In our specification formalism, deciding weak refinement is easier than deciding thorough refinement even though both are in EXPTIME. Nevertheless, since these two refinements do not coincide, in general, a procedure to check weak refinement cannot be used to decide thorough refinement.

Observe that weak refinement has a syntactic definition very much like simulation for transition systems. On the other hand, thorough refinement is a semantic concept, just as trace inclusion for transition systems. It is well known that simulation and trace inclusion coincide for deterministic automata. Similarly, for MTSs it is known that TR coincides with modal refinement for deterministic objects. It is thus natural to define deterministic IMCs and check whether thorough and weak refinements coincide on these objects.

In our context, an IMC is deterministic if, from a given state, one cannot reach two states that share common atomic propositions.

**Definition 7 (Determinism).** *An IMC $I = \langle Q, q_0, \varphi, A, V \rangle$ is* deterministic *iff for all states $q, r, s \in Q$, if there exists a distribution $\sigma \in \varphi(q)$ such that $\sigma(r) > 0$ and $\sigma(s) > 0$, then $V(r) \neq V(s)$.*

Weak determinism ensures that two states reachable *with the same admissible distribution* always have different valuations. In a semantic interpretation this means that there exists no implementation of $I$, in which two states with the same valuation can be successors of the same source state.

One can also propose another, more syntactic definition of determinism:

**Definition 8 (Strong Determinism).** *Let $I = \langle Q, q_0, \varphi, A, V \rangle$ be an IMC. $I$ is* strongly deterministic *iff for all states $q, r, s \in Q$, if there exist a probability distribution $\sigma \in \varphi(q)$ such that $\sigma(r) > 0$ and a probability distribution $\rho \in \varphi(q)$ such that $\rho(s) > 0$, then $V(r) \neq V(s)$.*

Strong determinism differs from the notion of determinism presented in Def. 7 in that it requires that, from a given state $q$, one cannot possibly reach two states $r$ and $s$ with the same set of propositions, even using *two different distributions* (implementations). Checking weak determinism requires solving a cubic number of linear constraints: for each state check the linear constraint of the definition—one

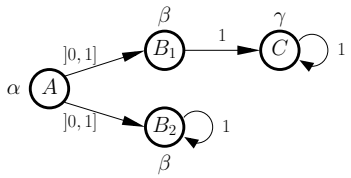



Figure 7: An IMC $I$ whose semantics cannot be captured by a deterministic IMC

per each pair of successors of a state. Checking strong determinism can be done by solving only a quadratic number of linear constraints—one per each successor of each state.

Luckily, due to the convexity of the set of admissible distributions in a state, these two notions coincide for IMCs, so the more efficient, strong determinism can be used in algorithms:

**Theorem 8.** *An IMC $I$ is deterministic iff it is strongly deterministic.*

*Proof.* It directly follows from the definitions that strong determinism implies weak determinism. We prove that if an IMC $I$ is not strongly deterministic, then it is not weakly deterministic either.

Let $I = \langle Q, q_0, \varphi, A, V \rangle$ be an IMC. If $I$ is not strongly deterministic, then there exist two admissible distributions on next states for $q$: $\sigma$ and $\rho \in \varphi(q)$ such that $\sigma(r) > 0$, $\sigma(s) = 0$, $\rho(r) = 0$, $\rho(s) > 0$ and $V(r) = V(s)$. In order to prove that $I$ is not weakly deterministic, we build a distribution $\gamma$ that we prove correct with respect to the interval specifications, i.e. $\gamma \in \varphi(q)$, and such that $\gamma(r) > 0$ and $\gamma(s) > 0$.

Since $\sigma(r) > 0$, there exists $a > 0$ such that $\varphi(q)(r) = [0, a]$ or $[0, a[$. Moreover, since $\rho(s) > 0$, there exists $b > 0$ such that $\varphi(q)(s) = [0, b]$ or $[0, b[$. Let $c = \mathsf{Min}(a, b)$, and define $\gamma(q') = \sigma(q')$ for all $q' \notin \{r, s\}$, $\gamma(r) = \sigma(r) - c/2$, and $\gamma(s) = c/2$. By construction, $\gamma \in \varphi(q)$ and we have $\gamma(r) > 0$ and $\gamma(s) > 0$. As a consequence, $I$ is not weakly deterministic. Finally, an IMC $I$ is strongly deterministic iff it is also weakly deterministic.

□

It is worth mentioning that deterministic IMCs are a strict subclass of IMCs. Figure 7 shows an IMC $I$ whose set of implementations cannot be represented by a deterministic IMC.

We now state the main theorem of the section that shows that for deterministic IMCs, the weak refinement, and indeed also the strong refinement, correctly

capture the thorough refinement:

**Theorem 9.** *For deterministic IMCs $I$ and $I'$ with no inconsistent states, the following statements are equivalent,*

1. *$I$ thoroughly refines $I'$,*
2. *$I$ weakly refines $I'$, and*
3. *$I$ strongly refines $I'$.*

*Proof.* It directly follows the definitions that (3) implies (2) and (2) implies (1). We will prove that (1) implies (2), and then that (2) implies (3).

Let $I_1 = \langle Q^1, q_0^1, \varphi_1, A, V_1 \rangle$ and $I_2 = \langle Q^2, q_0^2, \varphi_2, A, V_2 \rangle$ be two consistent and deterministic IMCs such that $[\![I_1]\!] \subseteq [\![I_2]\!]$.
First, remark that it is safe to suppose that implementations have the same set of atomic propositions as $I_1$ and $I_2$.

1. Let $\mathcal{R} \subseteq Q^1 \times Q^2$ be such that $r \,\mathcal{R}\, s$ iff for all MC $C$ and state $p$ of $C$, $p \models r \Rightarrow p \models s$. Since we consider pruned IMCs, there exist implementations for all states.
   Consider $r$ and $s$ such that $r \,\mathcal{R}\, s$.
   (a) By definition of $\mathcal{R}$, there exists a MC $C$ and a state $p$ of $C$ such that $p \models r$ and $p \models s$. Thus $V_C(p) = V_1(r)$ and $V_C(p) = V_2(s)$. As a consequence, $V_1(r) = V_2(s)$.
   (b) Consider $\rho \in \varphi_1(r)$ and build the MC $C = \langle Q^1, q_0^1, \pi, A, V_C \rangle$ such that for all $q \in Q^1$,
   - $V_C(q) = V_1(q)$;
   - If $q \neq r$, $\pi(q)$ is any distribution in $\varphi_1(q)$. At least one exists because $I_1$ is pruned;
   - $\pi(r) = \rho$.

   When necessary, we will address state $q$ of $C$ as $q_C$ to differentiate it from state $q$ of $I_1$. We will now build the correspondence function $\delta$.
   $C$ clearly satisfies $I_1$ with a satisfaction relation $\mathcal{R}_1 = \mathsf{Identity}$, and $r_C \models r$. By hypothesis, we thus have $r_C \models s$. Consider $\mathcal{R}_2$ the satisfaction relation such that $r_C \,\mathcal{R}_2\, s$ and $\delta_2$ the corresponding correspondence function. Let $\delta = \delta_2$.
   (c) As a consequence,
      i. By construction of $\delta$, we have that for all $q \in Q^1$, $\delta(q)$ is a probability distribution;

ii. By definition of the *s*atisfaction relation $\mathcal{R}_2$, we have that for all $s' \in Q^2$,
$\sum_{q_C \in Q^1} \rho(q_C)\delta_2(q_C)(s') \in \varphi_2(s)(s')$. As a consequence, for all $s' \in Q^2$, $\sum_{q \in Q^1} \rho(q)\delta(q)(s') \in \varphi_2(s)(s')$.

2. Let $r' \in Q^1$ and $s' \in Q^2$ be such that $\delta_{r's'} \neq 0$. By definition of $C$ and $\delta$, we have $r'_C \models r'$ and $r'_C \models s'$. We want to prove that for all implementations $C'$ and state $p'$ in $C'$, $p' \models r'$ implies $p' \models s'$.

   Suppose that this is not the case. There exists an implementation $C' = \langle P, o, \pi', A, V' \rangle$ and a state $p'$ of $C'$ such that $p' \models r'$ and $p' \not\models s'$. Let $\mathcal{R}'$ be the satisfaction relation witnessing $p' \models r'$.
   Consider the MC $\widehat{C} = \langle \widehat{Q^1} \cup \widehat{P}, \widehat{q_0^1}, \widehat{\pi}, A, \widehat{V} \rangle$. Intuitively, $\widehat{Q^1}$ corresponds to $C$ and $\widehat{P}$ corresponds to $C'$. The state $r'_C$ (called $\widehat{r'}$ in $\widehat{C}$) will be the link between the two and its outgoing transitions will be the ones of $p'$. Define

   - $\widehat{\pi}(\widehat{q_1})(\widehat{q_2}) = \pi(q_1)(q_2)$ if $q_1, q_2 \in Q^1$ and $\widehat{q_1} \neq \widehat{r'}$;
   - $\widehat{\pi}(\widehat{r'})(q_2) = 0$ if $q_2 \in Q^1$;
   - $\widehat{\pi}(\widehat{q_1})(\widehat{p_2}) = 0$ if $q_1 \in Q^1$ and $\widehat{q_1} \neq \widehat{r'}$ and $p_2 \in \widehat{P}$;
   - $\widehat{\pi}(\widehat{r'})(\widehat{p_2}) = \pi'(p')(p_2)$ if $p_2 \in P$;
   - $\widehat{\pi}(\widehat{p_1})(\widehat{q_2}) = 0$ if $p_1 \in P$ and $q_2 \in Q^1$;
   - $\widehat{\pi}(\widehat{p_1})(\widehat{p_2}) = \pi'(p_1)(p_2)$ if $p_1, p_2 \in P$;
   - $\widehat{V}(\widehat{q}) = V_1(q)$ if $q \in Q^1$;
   - $\widehat{V}(\widehat{p_1}) = V'(p_1)$ if $p_1 \in P$.

   We want to prove that $\widehat{r'}$ satisfies $s'$. This should imply that $p'_{C'}$ also satisfies $s'$, which is absurd.
   Consider the relation $\widehat{\mathcal{R}}$ between the states of $\widehat{C}$ and the states of $I_1$ defined as follows :

$$\begin{aligned}\widehat{\mathcal{R}} =& \{(\widehat{q^1}, q^{1'}) \mid (q_C^1, q^{1'}) \in R_1 \text{ and } \widehat{q^1} \neq \widehat{r'}\} \cup \\ & \{(\widehat{p^1}, q^{1'}) \mid (p^1, q^{1'}) \in \mathcal{R}'\} \cup \\ & \{(\widehat{r'}, q^{1'}) \mid p' \,\mathcal{R}'\, q^{1'}\}\end{aligned}$$

   Intuitively, $\widehat{\mathcal{R}}$ is equal to $\mathcal{R}_1$ for the states $\widehat{q^1} \in \widehat{Q^1}$, except $\widehat{r'}$, and equal to $\mathcal{R}'$ for the states $\widehat{p^1} \in \widehat{P}$. The states related to $\widehat{r'}$ are the ones that were related to $p'$ with $\mathcal{R}'$.

We will show that $\widehat{\mathcal{R}}$ is a satisfaction relation between $\widehat{C}$ and $I_1$.
Let $t, w$ be such that $t\widehat{\mathcal{R}}w$. For all the pairs where $t \neq \widehat{r'}$, the conditions of the satisfaction relation obviously still hold because they held for $\mathcal{R}_1$ if $t \in \widehat{Q^1}$ and for $\mathcal{R}'$ otherwise. It remains to check the conditions for the pairs where $t = \widehat{r'}$.
Consider $w$ such that $\widehat{r'}\widehat{\mathcal{R}}w$.

(a) Since $r'_C$ and $p'_{C'}$ are both implementations of $r'$, it is clear that $\widehat{V}(\widehat{r'}) = \widehat{V}(p')$. As $p' \mathcal{R}' w$, we know that $V'(p') = V_1(w)$. Thus, $\widehat{V}(\widehat{r'}) = V_1(w)$.

(b) Consider the correspondence function $\delta' : P \to (Q^1 \to [0,1])$ given by $p' \mathcal{R}' w$. Let $\widehat{\delta} : (\widehat{Q^1} \cup \widehat{P}) \to (Q^1 \to [0,1])$ be such that $\widehat{\delta}(\widehat{p^1}) = \delta'(p^1)$ whenever $\widehat{p^1} \in \widehat{P}$. Obviously, this is still a probability distribution on $Q^1$, and it is such that

i. for all $q^1 \in Q^1$,

$$\sum_{t \in \widehat{Q^1} \cup \widehat{P}} \widehat{\pi}(r')(t)\widehat{\delta}(t)(q^1) = \sum_{\widehat{p_2} \in \widehat{P}} \pi'(p')(p_2)\widehat{\delta}(\widehat{p_2})(q^1)$$
$$= \sum_{p_2 \in P} \pi'(p')(p_2)\delta'(p_2)(q^1).$$

By definition of $\delta'$, this is contained in $\varphi_1(w)(q^1)$.

ii. Moreover, if $\widehat{\pi}(\widehat{r'})(t) \neq 0$ and $\widehat{\delta}(t)(q^1) \neq 0$, then $t\widehat{\mathcal{R}}q^1$. We only need to consider $t = \widehat{p_1} \in \widehat{P}$ (since otherwise $\widehat{\pi}(\widehat{r'})(t) = 0$) and $q^1$ such that $\widehat{\delta}(\widehat{p_1})(q^1) \neq 0$. In this case, $\delta'(p_1)(q^1) \neq 0$. As $\delta'$ is a witness of $p' \mathcal{R}' w$, it has to be that $p_1 \mathcal{R}' q^1$, which implies, by definition of $\widehat{\mathcal{R}}$, that $t\widehat{\mathcal{R}}q^1$.

Finally, $\widehat{C}$ satisfies $I_1$, and in particular, $\widehat{r} \models r$. As $r \mathcal{R} s$, it implies that $\widehat{r} \models s$. As a consequence, there exists $\delta'' : (\widehat{Q^1} \cup \widehat{P}) \to (Q^2 \to [0,1])$ such that, for all $q^2 \in Q^2$,

$$\sum_{t \in \widehat{Q^1} \cup \widehat{P}} \widehat{\pi}(\widehat{r})(t)\delta''(t)(q^2) \in \varphi_2(s)(q^2)$$

(A) Consider $q^2 \neq s'$ such that $V_2(q^2) = V_2(s')$. Due to determinism of $I_2$, and to the fact that $s'$ is accessible from $s$, we have $\varphi_2(s)(q^2) = \{0\}$. Since $\widehat{\pi}(\widehat{r})(\widehat{r'}) \neq 0$ and $\widehat{\pi}(\widehat{r})(\widehat{r'})\delta''(\widehat{r'})(q^2)$ is part of the sum above, we must have $\delta''(\widehat{r'})(q^2) = 0$.

(B) Consider $q^3$ such that $V_2(q^3) \neq V_2(s') = V_1(r')$. It is clear that $\delta''(\widehat{r'})(q^3) = 0$ since $\delta''$ is witnessing satisfaction between $\widehat{C}$ and $I_2$.
(C) Moreover, since $\widehat{\pi}(\widehat{r})(\widehat{r'}) > 0$, we know that $\delta''(\widehat{r'})$ is a probability distribution over $Q^2$.

According to (A) and (B), the only non-zero value in the distribution in (C) must be $\delta''(\widehat{r'})(s')$. Since $\delta''$ is witnessing $\widehat{C} \models I_2$, this means that $\widehat{r'} \models s'$. By construction, $\widehat{r'}$ and $p'$ only differ by state names. This contradicts the assumption that $p' \not\models s'$. Thus $r' \mathcal{R} s'$, and $\mathcal{R}$ is a weak refinement relation.

Finally, we have by hypothesis that $[\![I_1]\!] \subseteq [\![I_2]\!]$, which implies that $q_0^1 \mathcal{R} q_0^2$. We thus have (1) implies (2).

□

We now prove that (2) implies (3). The following lemma is a direct consequence of determinism. It states that correspondence functions associated to a satisfaction relation for a deterministic IMC are of a particular form.

**Lemma 10.** *Let $I = \langle Q, q_0, \varphi, A, V\rangle$ be a deterministic IMC. Let $C = \langle P, p_0, \pi, A, V_C\rangle \in [\![I]\!]$ be a MC and let $\mathcal{R}$ be a satisfaction relation such that $p_0 \mathcal{R} q_0$. Let $p \in P$ and $q \in Q$ be such that $p \mathcal{R} q$, and let $\delta$ be the associated correspondence function. We have*

$$\forall p' \in P,\ \pi(p)(p') \neq 0 \Rightarrow |\{q' \in Q \mid \delta(p')(q') \neq 0\}| = 1. \tag{2}$$

Obviously, the same holds for correspondence functions associated to refinement relations between deterministic IMCs.

Let $I_1 = \langle Q^1, q_0^1, \varphi_1, A, V_1\rangle$ and $I_2 = \langle Q^2, q_0^2, \varphi_2, A, V_2\rangle$ be two deterministic IMCs such that $I_1 \preceq_W I_2$ with a weak refinement relation $\mathcal{R}$. We prove that $\mathcal{R}$ is in fact a strong refinement relation.

Let $p \in Q^1$ and $q \in Q^2$ be such that $p \mathcal{R} q$.

1. By hypothesis, $V_1(p) = V_2(q)$;
2. We know that for all probability distribution $\sigma \in \varphi_1(p)$, there exists a correspondence function $\delta^\sigma$ satisfying the axioms of a (weak) refinement relation. We will build a correspondence function $\delta^0$ that will work for all $\sigma$. Let $p' \in Q^1$.
   - If for all $\sigma \in \varphi_1(p)$, we have $\sigma(p') = 0$, then let $\delta^0(p', q') = 0$ for all $q' \in Q^2$;

- Else, consider $\sigma \in \varphi_1(p)$ such that $\sigma(p') \neq 0$. By hypothesis, there exists a correspondence function $\delta^\sigma$ associated to $p \mathcal{R} q$. Let $\delta^0(p') = \delta^\sigma(p')$. By Lemma 10, there is a single $q' \in Q^2$ such that $\delta^\sigma(p')(q') \neq 0$. Moreover, by definition of $\delta^\sigma$, we know that $\sum_{q'' \in Q^2} \delta^\sigma(p')(q'') = 1$, thus $\delta^\sigma(p')(q') = 1$.

  Suppose there exists $\rho \neq \sigma \in \varphi_1(p)$ such that $\rho(p') \neq 0$. Let $\delta^\rho$ be the associated correspondence function. As for $\sigma$, there exists a unique $q'' \in Q^2$ such that $\delta^\rho(p')(q'') \neq 0$. Moreover $\delta^\rho(p')(q'') = 1$. By definition of $\delta^\sigma$ and $\delta^\rho$, we have

$$\mu : q''' \mapsto \sum_{p'' \in Q^1} (\sigma(p'')\delta^\sigma(p'')(q''')) \in \varphi_2(q)$$
$$\nu : q''' \mapsto \sum_{p'' \in Q^1} (\rho(p'')\delta^\rho(p'')(q''')) \in \varphi_2(q)$$

  Moreover, both $\mu(q') > 0$ and $\nu(q'') > 0$. By determinism of $I_2$, this implies $q' = q''$.

  As a consequence, we have $\delta^\sigma(p') = \delta^\rho(p')$, so $\forall \gamma \in \varphi_1(p)$, if $\gamma(p') > 0$, then $\delta^\gamma(p') = \delta^0(p')$.

  Finally, consider $\delta^0$ defined as above. Let $\sigma \in \varphi_1(p)$. We have
  (a) if $\sigma(p') > 0$, then $\delta^0(p') = \delta^\sigma(p')$ is a distribution over $Q^2$;
  (b) for all $q' \in Q^2$,

$$\sum_{p' \in Q^1} (\sigma(p')\delta^0(p')(q')) = \sum_{p' \in Q^1} (\sigma(p')\delta^\sigma(p')(q')) \in \varphi_2(q)(q') \text{ by definition of } \delta^\sigma;$$

  (c) if $\delta^0(p')(q') > 0$, then there exists $\sigma \in \varphi_1(p)$ such that $\delta^0(p')(q') = \delta^\sigma(p'q') > 0$, thus $p' \mathcal{R} q'$ by definition of $\delta^\sigma$.

  Finally, $\mathcal{R}$ is a strong refinement relation.

□

## 6. Common Implementation and Consistency

We now turn our attention to the problem of implementation of several IMC specifications by the same probabilistic system modeled as a Markov Chain. We start with defining the problem:

**Definition 9 (Common Implementation (CI)).** *Given $k > 1$ IMCs $I_i$,$i = 1 \dots k$, does there exist a Markov Chain $C$ such that $C \models I_i$ for all $i$?*

Somewhat surprisingly we find out that, similarly to the case of TR, the CI problem is not harder for IMCs than for modal transition systems:

**Theorem 11.** *Deciding the existence of a CI between $k$ IMCs is EXPTIME-complete in general.*

**Lower Bound.** To establish a lower bound for common implementation, we propose a reduction from the common implementation problem for modal transition systems (MTS). This latter problem has recently been shown to be EXPTIME-complete when the number of MTS is not known in advance and PTIME-complete otherwise [16]. We first propose the following theorem.

**Theorem 12.** *Let $M_i$ be MTSs for $i = 1, \dots, k$. We have*

$$\exists I \forall i : I \models M_i \iff \exists C \forall i : C \models \widehat{M_i},$$

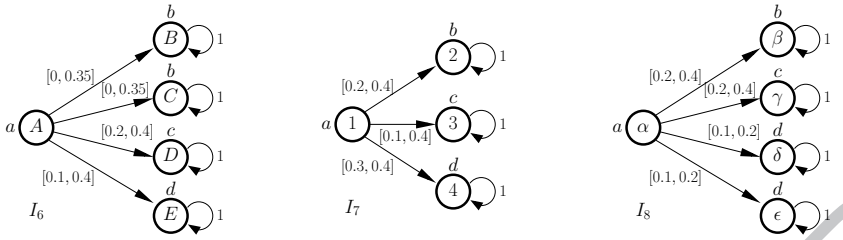
*where $I$ is a transition system, $C$ is a Markov Chain and $\widehat{M_i}$ is the IMC obtained with the transformation defined in Section 4.*

*Proof.* $\Rightarrow$: This direction can be proven by showing that for arbitrary $j \in \{1, \dots, k\}$, $[\![\widehat{I}]\!] \subseteq [\![\widehat{M_j}]\!]$. This is indeed the result of Theorem 6. Now pick a $C \in [\![\widehat{I}]\!]$, and the result follows.
$\Leftarrow$: Assume that there exists a $C$ such that $C \models \widehat{M_i}$ for all $i = 1, \dots, k$. With the transformation defined in section 4, an implementation $I$ for all $M_i$ for all $i$ can be constructed as $f(C)$. □

**Upper Bound.** To address the upper bound we first propose a simple construction to check if there exists a CI for two IMCs. We start with the definition of *consistency relation* that witnesses a common implementation between two IMCs.

**Definition 10.** *Let $I_1 = \langle Q_1, q_0^1, \varphi_1, A, V_1 \rangle$ and $I_2 = \langle Q_2, q_0^2, \varphi_2, A, V_2 \rangle$ be IMCs. Then $\mathcal{R} \subseteq Q_1 \times Q_2$ is a* consistency relation *on the states of $I_1$ and $I_2$ iff whenever $(u, v) \in \mathcal{R}$ then*

- $V_1(u) = V_2(v)$,
- *there exists a $\rho \in \mathsf{Distr}(Q_1 \times Q_2)$ such that*




(a) $I_6$ (b) $I_7$ (c) $I_8$

Figure 8: IMCs $I_6$, $I_7$, and $I_8$

1. $\forall u' \in Q_1 : \sum_{v' \in Q_2} \rho(u', v') \in \varphi_1(u)(u') \wedge \forall v' \in Q_2 : \sum_{u' \in Q_1} \rho(u', v') \in \varphi_2(v)(v')$, *and*
2. $\forall (u', v') \in Q_1 \times Q_2$ *st.* $\rho(u', v') > 0$, *then* $(u', v') \in \mathcal{R}$.

We illustrate the definition of a consistency relation in the following example.

**Example 2.** *Consider the three IMCs in Figure 8. We construct a consistency relation $\mathcal{R}$ for $k = 3$. The triple $(A, 1, \alpha)$ is in the relation $\mathcal{R}$ witnessed by the distribution $\rho$ that assigns $\frac{1}{6}$ to $(B, 2, \beta)$, $\frac{1}{6}$ to $(C, 2, \beta)$, $\frac{1}{3}$ to $(D, 3, \gamma)$, $\frac{1}{6}$ to $(E, 4, \delta)$, and $\frac{1}{6}$ to $(E, 4, \epsilon)$. The triples that are given positive probability by $\rho$ are also in the relation each by the distribution assigning probability 1 to itself. A common implementation $C = \langle P, p_0, \pi, A, V_C \rangle$ can be constructed as follows: $P = \{q | q \in \mathcal{R}\}$, $p_0 = (A, 1, \alpha)$, $V_C(p)$ is inherited from $I_6$, $I_7$, and $I_8$, and $\pi(p)(p') = \rho(p')$, where $\rho$ is the distribution witnessing that $p \in \mathcal{R}$.*

We now prove that the existence of a consistency relation is equivalent to the existence of a common implementation, in the case of $k = 2$. The above definition and the following theorem extends to general $k$.

**Theorem 13.** *Let $I_1 = \langle Q_1, q_0^1, \varphi_1, A, V_1 \rangle$ and $I_2 = \langle Q_2, q_0^2, \varphi_2, A, V_2 \rangle$ be IMCs. $I_1$ and $I_2$ have a common implementation iff there exists a consistency relation $\mathcal{R}$ such that $q_0^1 \, \mathcal{R} \, q_0^2$.*

*Proof.* $\Rightarrow$: Assume that there exists a MC $C = \langle P, p_0, \pi, A, V_C \rangle$ such that $C \models I_1$ and $C \models I_2$. This implies that there exists satisfaction relations $\mathcal{R}_1 \subseteq P \times Q_1$ and $\mathcal{R}_2 \subseteq P \times Q_2$ such that $p_0 \, \mathcal{R}_1 \, q_0^1$ and $p_0 \, \mathcal{R}_2 \, q_0^2$.

A relation $\mathcal{R}$ is constructed as $\{(q_1, q_2) | \exists p \in P : p \, \mathcal{R}_1 \, q_1 \wedge p \, \mathcal{R}_2 \, q_2\}$. We now prove that $\mathcal{R}$ is a consistency relation relating $q_0^1$ and $q_0^2$; indeed $(q_0^1, q_0^2) \in \mathcal{R}$ because $p_0 \, \mathcal{R}_1 \, q_0^1$ and $p_0 \, \mathcal{R}_2 \, q_0^2$. Let $(q_1, q_2) \in \mathcal{R}$ and $p \in P$ be such that $p \, \mathcal{R}_1 \, q_1$ and $p \, \mathcal{R}_2 \, q_2$.

1. By $\mathcal{R}_1$ and $\mathcal{R}_2$, $V_1(q_1) = V_C(p) = V_2(q_2)$
2. Let $\delta_1$ and $\delta_2$ be the correspondence functions witnessing $p \,\mathcal{R}_1\, q_1$ and $p \,\mathcal{R}_2\, q_2$, and let $\rho \in \mathsf{Distr}(Q_1 \times Q_2)$ be such that

$$\rho(q_1', q_2') = \sum_{p' \in P \text{ st. } \pi(p)(p') > 0} \pi(p)(p')\delta_1(p', q_1')\delta_2(p', q_2'). \tag{3}$$

   Since $\sum_{q_1' \in Q_1} \sum_{q_2' \in Q_2} \rho(q_1', q_2') = 1$, $\rho$ is indeed a distribution on $Q_1 \times Q_2$. Let $u' \in Q_1$.

$$\begin{aligned}
\sum_{v' \in Q_2} \rho(u', v') &= \sum_{(v' \in Q_2)} \sum_{(p' \in P \text{ st. } \pi(p)(p') > 0)} \pi(p)(p')\delta_1(p', u')\delta_2(p', v') \\
&= \sum_{p' \in P \text{ st. } \pi(p)(p') > 0} \pi(p)(p')\delta_1(p', u') \sum_{v' \in Q_2} \delta_2(p', v') \\
&= \sum_{p' \in P \text{ st. } \pi(p)(p') > 0} \pi(p)(p')\delta_1(p', u') \quad \text{by definition of } \delta_2 \\
&\in \varphi_1(q_1)(u') \quad \text{by definition of } \delta_1.
\end{aligned}$$

   Similarly, for all $v' \in Q_2$, $\sum_{u' \in Q_1} \rho(u', v') \in \varphi_2(v)(v')$.
3. Let $q_1' \in Q_1$ and $q_2' \in Q_2$ be states such that $\rho(q_1', q_2') > 0$. Then at least one term in Eq. (3) is positive. Thus, there exists $p'$ such that

$$\pi(p)(p')\delta_1(p', q_1')\delta_2(p', q_2') > 0.$$

   This implies that all factors are positive, and by definition of $\delta_1$ and $\delta_2$, we have that $(p', q_1') \in \mathcal{R}_1$ and $(p', q_2') \in \mathcal{R}_2$ and therefore $q_1' \,\mathcal{R}\, q_2'$.

This proves that $\mathcal{R}$ is a consistency relation.

$\Leftarrow$: Assume that there exists a consistency relation $\mathcal{R}$ relating $q_0^1$ and $q_0^2$. We now construct a common implementation $C$, such that $C \models I_1$ and $C \models I_2$; we prove the former first. Let $C = \langle P, p_0, \pi, A, V_C \rangle$ be such that

- $P = \{(q_1, q_2) \in Q_1 \times Q_2 \mid q_1 \,\mathcal{R}\, q_2\}$
- $p_0 = (q_0^1, q_0^2)$
- $V_C((q_1, q_2)) = V_1(q_1) = V_2(q_2)$ by definition of $\mathcal{R}$
- For each $(q_1, q_2), (q_1', q_2') \in P$, $\pi((q_1, q_2)(q_1', q_2')) = \rho(q_1', q_2')$, where $\rho$ is the distribution witnessing the membership of $(q_1, q_2)$ in $\mathcal{R}$.

To show satisfaction between $C$ and $I_1$, the relation $\mathcal{R}_s$ is used. It is defined as follows: for all $(u, v) \in P$, $(u, v)\,\mathcal{R}_s\,w$ iff $u = w$. We now show that $\mathcal{R}_s$ is a satisfaction relation between $C$ and $I_1$.
Let $(u, v) \in P$ be such that $(u, v)\,\mathcal{R}_s\,u$.

1. By definition of $C$, $V_C(u, v) = V_1(u)$
2. Let $\delta$ be the correspondence function such that: $\delta((u', v'), q_1) = 1$ if $u' = q_1$ and 0 else.
   (a) Let $(u', v') \in P$ be such that $\pi(u, v)(u', v') > 0$. $\delta((u', v'))$ is a distribution by definition.
   (b) Let $q_1 \in Q_1$.

$$\begin{aligned}\sum_{(u',v')\in P} \pi(u,v)(u',v')\delta((u',v'),q_1) &= \sum_{(q_1,v')\in P} \pi((u,v),(q_1,v')) \\ &= \sum_{v'\in Q_2} \rho(q_1,v') \\ &\in \varphi_1(u)(q_1) \quad \text{by definition of } \mathcal{R}\,.\end{aligned}$$

   (c) Let $(u', v') \in P$ and $q_1 \in Q_1$ be such that $\delta((u', v'), q_1) > 0$. Then $u' = q_1$ and by definition, $(u', v')\,\mathcal{R}_s\,q_1$.

Consequently, $\mathcal{R}_s$ is a satisfaction relation, and thus $C \models I_1$. Analogously, it can be shown that $C \models I_2$. Finally $C$ is a common implementation of $I_1$ and $I_2$. □

As a consequence, deciding the existence of a common implementation between 2 IMCs is PTIME-complete. For the general problem of common implementation of $k$ IMCs, we can extend the above definition of consistency relation to the $k$-ary relation in the obvious way, and the algorithm becomes exponential in the number of IMCs $k$, as the size of the state space $\prod_{i=1}^{k} |Q_i|$ is exponential in $k$.

As a side effect we observe that, exactly like MTSs, CI becomes polynomial for any constant value of $k$, i.e. when the number of components to be checked is bounded by a constant.

*Consistency.* A related problem is the one of checking consistency of a single IMC $I$, i.e. whether there exists a Markov chain $M$ such that $M \models I$.

**Definition 11 (Consistency (C)).** *Given an IMC $I$, does it hold that $[\![I]\!] \neq \emptyset$?*

It turns out that, in the complexity theoretic sense, this problem is easy:

**Theorem 14.** *The problem C, to decide if a single IMC is consistent, is polynomial time solveable.*

*Proof.* Given an IMC $I = \langle Q, q_0, \varphi, A, V \rangle$, this problem can be solved by constructing a consistency relation over $Q \times Q$ (as if searching for a common implementation of $Q$ with itself). Now there exists an implementation of $I$ iff there exists a consistency relation containing $(q_0, q_0)$. Obviously, this can be checked in polynomial time. □

The fact that C can be decided in polynomial time casts an interesting light on the ability of IMCs to express inconsistency. On one hand, one can clearly specify inconsistent states in IMCs (simply by giving intervals for successor probabilities that cannot be satisfied by any distribution). On the other hand, this inconsistency appears to be local. It does not induce any global constraints on implementations; it does not affect consistency of other states. In this sense IMCs are weaker than *mixed transition systems* [20]. Mixed transition systems relax the requirement of modal transition systems, not requiring that $(\rightarrow) \subseteq (\dashrightarrow)$. It is known that C is trivial for modal transition systems, but EXPTIME-complete for mixed transition systems [16]. Clearly, with a polynomial time C, IMCs cannot possibly express global behaviour inconsistencies in the style of mixed transition systems, where the problem is much harder.

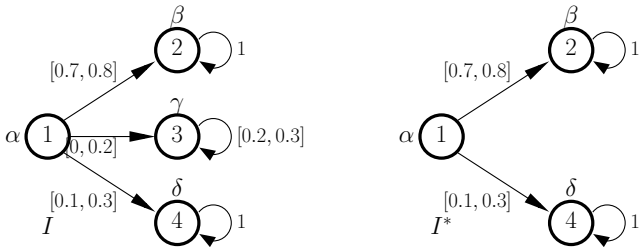
We conclude the section by observing that, given the IMC $I$ and a consistency relation $\mathcal{R} \subseteq Q \times Q$, it is possible to derive a *pruned* IMC $I^* = \langle Q^*, q_0^*, \varphi^*, A, V^* \rangle$ that contains no inconsistent states and accepts the same set of implementations as $I$.

The construction of $I^*$ is as follows: $Q^* = \{q \in Q | (q, q) \in \mathcal{R}\}$, $q_0^* = q_0$, $V^*(q^*) = V(q^*)$ for all $q^* \in Q^*$, and for all $q_1^*, q_2^* \in Q^*$, $\varphi^*(q_1^*)(q_2^*) = \varphi(q_1^*)(q_2^*)$.

**Theorem 15.** *Consider an IMC $I$ and its pruned IMC $I^*$. It holds that $[\![I]\!] = [\![I^*]\!]$.*

*Proof.*

1. We first prove that $[\![I]\!] \subseteq [\![I^*]\!]$. Let $\mathcal{R} \subseteq Q \times Q$ be a consistency relation such that $(q_0, q_0) \in \mathcal{R}$, and let $C = \langle P, p_0, \pi, A, V_C \rangle$ be a MC such that $C \models I$ with satisfaction relation $\mathcal{R}_s$. We build a satisfaction relation $\mathcal{R}'_s \subseteq P \times Q^*$ where $p \,\mathcal{R}'\, q^*$ iff there exists $q \in Q$ such that $p \,\mathcal{R}_s\, q$ and $q = q^*$. Let $p \in P$, $q \in Q$, and $q^* \in Q^*$ be such that $(p, q^*) \in \mathcal{R}'$. We now show that $\mathcal{R}'$ is a satisfaction relation between $P$ and $I^*$.
   - By construction, $V_C(p) = V^*(q^*)$.




(a) IMC $I$ (b) Pruned IMC $I^*$

Figure 9: An IMC and its pruned version

- Let $\delta_1 \in \mathsf{Distr}(P \times Q)$ be the distribution witnessing $p \mathcal{R}_s q$. The distribution $\delta_2 \in \mathsf{Distr}(P \times Q^*)$ is chosen identical to $\delta_1$. We know that for all $q' \in Q$ such that $\neg\exists\sigma \in \varphi(q')$ then for all $p' \in P$, we have that $\delta_1(p', q') = 0$. To see this, assume the contrary, namely that $\delta_1(p', q') \neq 0$ for a $p' \in P$ and a $q' \in Q$ for which $\neg\exists\sigma \in \varphi(q')$; then $p' \mathcal{R}_s q'$. By the definition af satisfaction, $q'$ allows a distribution, which is a contradiction.

  Since $\delta_1$ satisfies the axioms of satisfaction, then $\delta_2$ also satisfies them.

2. To show that $[\![I^*]\!] \subseteq [\![I]\!]$, we use the same reasoning as above.

By mutual inclusion, $[\![I]\!] = [\![I^*]\!]$. □

An illustration of pruning is given in the following example.

**Example 3.** *Consider the IMC $I$ in Figure 9a. Building a consistency relation, we see that $(1, 1)$ is in the relation witnessed by the distribution assigning probability $0.8$ to $(2, 2)$ and $0.2$ to $(4, 4)$. This probability distribution "avoids" the inconsistent state $(3, 3)$; this state does not admit a probability distribution. Likewise, $(2, 2)$ and $(3, 3)$ are in the relation, witnessed by the distributions that gives probability 1 to $(2, 2)$ and $(3, 3)$, respectively. $I^*$ is shown in Figure 9b.*

## 7. Conclusion and Future Work

This paper provides new results for IMCs [1, 21, 22, 23] that is a specification formalism for probabilistic systems. We have studied the expressiveness and complexity of three refinement preorders for IMCs. The results are of interest

as existing articles on IMCs often use one of these preorders to compare specifications (for abstractions) [1, 13, 12]. We have established complexity bounds and decision procedures for these relations, first introduced in [1]. Finally, we have studied the common implementation problem that is to decide whether there exists an implementation that can match the requirements made by two or more specifications. Our solution is constructive in the sense that it can build such a common implementation.

Our results are robust with respect to simple variations of IMCs. For example sets of sets of propositions can be used to label states, instead of sets of propositions. This extends the power of the modeling formalism, which now can not only express abstractions over probability distributions, but also over possible state valuations. Similarly, an initial distribution, or even an interval constraint on the initial distribution, could be used instead of the initial state in IMCs without affecting the results.

In the future we expect to see whether our complexity results can be extended to CMCs [2]—an already mentioned generalization of IMCs, which enjoys good closure properties. Furthermore, in order to improve efficiency of tools, it would be desirable to investigate whether IMCs could be used as an abstraction in counter-example guided abstraction-refinement [24] decision procedures for CMCs.

In [13, 25], Katoen et al. have proposed an extension of IMCs to the continuous timed setting. It would be interesting to see whether our results extend to this new model. Another interesting future work would be to extend our results to other specification formalisms for systems that mix both stochastic and nondeterministic aspects. Among them, one finds probabilistic automata [26] where weak/strong refinement would be replaced by probabilistic simulation [27].

Markov set-chains allow iterative approximation of implementations with increasing state space size. It would be interesting to investigate if these could be used to define size-parameterized versions of our decision problems, and whether these could be solved by iterative approximations.

## References

[1] B. Jonsson, K. G. Larsen, Specification and refinement of probabilistic processes, in: LICS, IEEE Computer, 1991, pp. 266–277.

[2] B. Caillaud, B. Delahaye, K. G. Larsen, A. Legay, M. L. Pedersen, A. Wąsowski, Compositional design methodology with constraint markov chains, in: QEST, IEEE Computer, 2010.

[3] B. Delahaye, K. G. Larsen, A. Legay, M. L. Pedersen, A. Wąsowski, New results for constraint markov chains, Submitted for review.
[4] N. Benes, J. Kretínský, K. G. Larsen, J. Srba, Checking thorough refinement on modal transition systems is exptime-complete, in: ICTAC, 2009, pp. 112–126.
[5] S. Andova, Process algebra with probabilistic choice, in: ARTS, Springer-Verlag, London, UK, 1999, pp. 111–129.
[6] N. López, M. Núñez, An overview of probabilistic process algebras and their equivalences, in: VSS, Vol. 2925 of LNCS, Springer, 2004, pp. 89–123.
[7] H. Hansson, B. Jonsson, A logic for reasoning about time and reliability, Formal Asp. Comput. 6 (5) (1994) 512–535.
[8] H. J. Hartfield, Markov Set-Chains, Vol. 1695 of Lecture Notes in Mathematics, Springer Verlag, 1998.
[9] A. Abate, A. D'Innocenzo, M. D. D. Benedetto, S. S. Sastry, Markov set-chains as abstractions of stochastic hybrid sytems, in: M. Egerstedt, B. Mishra (Eds.), Proceedings of the 11th international workshop on Hybrid Systems: Computation and Control, Vol. 4981 of LNCS, Springer Verlag, 2008.
[10] E. M. Clarke, O. Grumberg, D. E. Long, Model checking and abstraction, ACM Transactions on Programming Languages and Systems 16 (5) (1994) 1512–1542.
[11] E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, H. Veith, Counterexample-guided abstraction refinement for symbolic model checking, J. ACM 50 (5) (2003) 752–794.
[12] H. Fecher, M. Leucker, V. Wolf, Don't Know in probabilistic systems, in: SPIN, Vol. 3925 of LNCS, Springer, 2006, pp. 71–88.
[13] J. Katoen, D. Klink, M. Leucker, V. Wolf, Three-valued abstraction for continuous-time Markov chains, in: CAV, Vol. 4590 of LNCS, Springer, 2007, pp. 311–324.
[14] K. G. Larsen, B. Thomsen, A modal process logic, in: LICS, IEEE Computer Society, 1988, pp. 203–210.
[15] K. G. Larsen, Modal specifications, in: AVMS, Vol. 407 of LNCS, 1989, pp. 232–246.
[16] A. Antonik, M. Huth, K. G. Larsen, U. Nyman, A. Wąsowski, Modal and mixed specifications: key decision problems and their complexities, MSCS 20 (01) (2010) 75–103.
[17] A. Antonik, M. Huth, K. G. Larsen, U. Nyman, A. Wąsowski, 20 years of modal and mixed specifications, BEATCS 95, available at http://processalgebra.blogspot.com/2008/05/concurrency-column-for-beatcs-june-2008.html.
[18] B. Jonsson, K. G. Larsen, W. Yi, Probabilistic extensions of process algebras, in: Handbook of Process Algebra, Elsevier, 2001, pp. 685–710.
[19] M. R. Henzinger, T. A. Henzinger, P. W. Kopke, Computing simulations on finite and infinite graphs, in: Proc. FOCS'95, 1995, pp. 453–462.
[20] D. Dams, Abstract interpretation and partition refinement for model checking, Ph.D. thesis, Eindhoven University of Technology (July 1996).

[21] K. Sen, M. Viswanathan, G. Agha, Model-checking Markov chains in the presence of uncertainties, in: TACAS, Vol. 3920 of LNCS, Springer, 2006, pp. 394–410.

[22] K. Chatterjee, K. Sen, T. A. Henzinger, Model-checking omega-regular properties of interval Markov chains, in: FoSSaCS, Vol. 4962 of LNCS, Springer, 2008, pp. 302–317.

[23] S. Haddad, N. Pekergin, Using stochastic comparison for efficient model checking of uncertain Markov chains, in: QEST, IEEE, 2009, pp. 177–186.

[24] E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, H. Veith, Counterexample-guided abstraction refinement, in: E. A. Emerson, A. P. Sistla (Eds.), CAV, Vol. 1855 of Lecture Notes in Computer Science, Springer, 2000, pp. 154–169.

[25] J. Katoen, D. Klink, M. R. Neuhäußer, Compositional abstraction for stochastic systems, in: FORMATS, Vol. 5813 of LNCS, Springer, 2009, pp. 195–211.

[26] M. O. Rabin, Probabilistic automata, Inf. and Cont. 6 (3) (1963) 230–245.

[27] R. Segala, N. Lynch, Probabilistic simulations for probabilistic processes, in: CONCUR, Vol. 836 of LNCS, Springer, 1994, pp. 481–496.