Second Order Connectives and Proof Transformations in Linear Logic

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Abstract

This article presents a method for transforming linear logic proofs that reduces the number of primitive constructions. Inside formulas, some connectives are replaced by *second order* equivalents. Proofs are also transformed and cut elimination may be emulated. This mechanism leads to proofs without the additives & and $\oplus$ and the constants 1, $\perp$, 0 and $\top$ and where the modality ! is only introduced with empty environments. As CPS-transformations in λ-calculus leads to particular reduction strategies, those transformations inhibit some reductions known as *commutative conversions*. As a consequence, proof normalization is Church-Rosser in this system. This method captures strategies of normalization used by various translation from linear logic to interaction nets. It can be seen as an intermediate translation giving a theoretical justification for those coding. Another interesting aspect of *second order exponentials* is their similarities to combinatorial exponentials.

1 Introduction

Linear logic [Gir87], which may be seen as a logic that takes resources into account, has a sequent calculus presentation. Another presentation consists in using proof structures and proof nets composed of connectives and boxes linked together to form nets. The aim of this notion is to identify certain not conceptually different proofs in sequent calculus. However, proof nets are not perfect since we would like to identify more proofs than proof nets does. This is due to *boxes* which delimit the influence of certain context sensitive connectives ($\perp$, $\top$, & and !).

This article presents a way of eliminating the sensitivity of these connectives by using *second order* equivalents:

$$\begin{array}{lcl}
1' & \stackrel{\mathbf{Def}}{=} & \forall X.X^{\perp}\wp X \\
\perp' & \stackrel{\mathbf{Def}}{=} & \exists X.X^{\perp}\otimes X \\
0' & \stackrel{\mathbf{Def}}{=} & \forall X.X \\
\top' & \stackrel{\mathbf{Def}}{=} & \exists X.X \\
U\oplus' V & \stackrel{\mathbf{Def}}{=} & \forall X.?(X^{\perp}\otimes U)\wp ?(X^{\perp}\otimes V)\wp X \\
U\&'V & \stackrel{\mathbf{Def}}{=} & \exists X.!(X^{\perp}\wp U)\otimes !(X^{\perp}\wp V)\otimes X \\
?'U & \stackrel{\mathbf{Def}}{=} & \forall X.?(X^{\perp}\otimes \perp')\wp ?(X^{\perp}\otimes (X\wp X))\wp ?(X^{\perp}\otimes U)\wp X \\
!'U & \stackrel{\mathbf{Def}}{=} & \exists X.!(X^{\perp}\wp 1')\otimes !(X^{\perp}\wp (X\otimes X))\otimes !(X^{\perp}\wp U)\otimes X
\end{array}$$

The principle of these new constructions is to divide each context-sensitive connective in two parts, one managing the context and the other introducing the connective. This leads to formula transformations and proof transformations that preserve the elimination of main cuts. However, like CPS-transformations, this mechanism eliminates non-essential cuts known as commutative conversions [Gir87, GLT88]. A benefit of this strategy is to force cut elimination to be Church-Rosser.

In fact there are two motivations for those transformations. The first one is that this method inhibits commutative conversion. As a consequence, this strategy can be compared to codings of linear logic into interaction nets [Laf95] where cut-elimination is done only for main reductions. Those transformations can be seen as an intermediary step that gives a theoretical meaning of those codings. The second motivation concerns our *second order exponentials* that look like the combinatorial approach of linear logic exponentials.

This article presents successively the transformations for the additives $\oplus$ and $\&$, for exponentials and finish by constants.

2 Elimination of Additives

2.1 Second Order Additives

System F [GLT88] uses two kinds of sum type. The first version defines a new connective $+$, rules to introduce and eliminate it and conversions of redexes. The second version emulates it as the abbreviation $U+V \stackrel{\mathbf{Def}}{=} \forall X.(U \to X) \to (V \to X) \to X$.

Linear logic has the same property. We can use the additives $\oplus$ and $\&$ as primitive operators or we can define them as abbreviations.

Definition 1 *The second order additives are defined by:*

$$\begin{array}{rcl} U \oplus' V & \stackrel{\mathbf{Def}}{=} & \forall X.?(X^\perp \otimes U)\wp?(X^\perp \otimes V)\wp X \\ U \&' V & \stackrel{\mathbf{Def}}{=} & \exists X.!(X^\perp \wp U)\otimes!(X^\perp \wp V) \otimes X \end{array}$$

In linear logic, classical and second order additives are linearly equivalent. We can prove that $\&$ and $\&'$ (resp. $\oplus$ and $\oplus'$) imply each other.

Theorem 2 *The following sequents are provable:*

$$\begin{array}{rcl} U \oplus V & \vdash & U \oplus' V \\ U \oplus' V & \vdash & U \oplus V \end{array} \qquad \begin{array}{rcl} U \& V & \vdash & U \&' V \\ U \&' V & \vdash & U \& V \end{array}$$

Proof: Since proofs of the right sequents may be obtained by duality from proofs of the left ones, we only need to give proofs of $U \oplus V \vdash U \oplus' V$ and $U \oplus' V \vdash U \oplus V$. We have to prove the sequents $\vdash U^\perp \& V^\perp, U \oplus' V$ and $\vdash U^\perp \&' V^\perp, U \oplus V$:

$$\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}
\UnaryInfC{$\vdash U^\perp, U$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash U^\perp, X^\perp \otimes U, X$}
\RightLabel{$?$}
\UnaryInfC{$\vdash U^\perp, ?(X^\perp \otimes U), ?(X^\perp \otimes V), X$}
\RightLabel{$\wp$}
\UnaryInfC{$\vdash U^\perp, ?(X^\perp \otimes U)\wp?(X^\perp \otimes V)\wp X$}
\RightLabel{$\forall X$}
\UnaryInfC{$\vdash U^\perp, \forall X.?(X^\perp \otimes U)\wp?(X^\perp \otimes V)\wp X$}
\AxiomC{}
\UnaryInfC{$\vdash X^\perp, X$}
\AxiomC{}
\UnaryInfC{$\vdash V^\perp, V$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash V^\perp, X^\perp \otimes V, X$}
\RightLabel{$?$}
\UnaryInfC{$\vdash V^\perp, ?(X^\perp \otimes U), ?(X^\perp \otimes V), X$}
\RightLabel{$\wp$}
\UnaryInfC{$\vdash V^\perp, ?(X^\perp \otimes U)\wp?(X^\perp \otimes V)\wp X$}
\RightLabel{$\forall X$}
\UnaryInfC{$\vdash V^\perp, \forall X.?(X^\perp \otimes U)\wp?(X^\perp \otimes V)\wp X$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash U^\perp \& V^\perp, \forall X.?(X^\perp \otimes U)\wp?(X^\perp \otimes V)\wp X$}
\RightLabel{$\mathbf{Def}$}
\UnaryInfC{$\vdash U^\perp \& V^\perp, U \oplus' V$}
\end{prooftree}$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash U, U^{\perp}}}{\vdash U \oplus V, U^{\perp}}\,1\oplus}{\vdash (U \oplus V) \wp U^{\perp}}\,\wp}{\vdash !((U \oplus V) \wp U^{\perp})}\,!
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash V, V^{\perp}}}{\vdash (U \oplus V), V^{\perp}}\,2\oplus}{\vdash (U \oplus V) \wp V^{\perp}}\,\wp}{\vdash !((U \oplus V) \wp V^{\perp})}\,!
\qquad
\dfrac{}{\vdash (U \oplus V)^{\perp}, U \oplus V}
}{
\dfrac{\dfrac{\vdash !((U \oplus V) \wp U^{\perp}) \otimes !((U \oplus V) \wp V^{\perp}) \otimes (U \oplus V)^{\perp}, U \oplus V}{\vdash \exists X.!(X^{\perp} \wp U^{\perp}) \otimes !(X^{\perp} \wp V^{\perp}) \otimes X, U \oplus V}\,\exists X = (U \oplus V)^{\perp}}{\vdash U^{\perp} \&' V^{\perp}, U \oplus V}\,\mathbf{Def}
}\,\otimes
$$

$\square$

Exchange rule has been omitted in the above proofs by considering a sequent as a multi-set of formulas.

2.2 Proof Transformations for Additives

The aim of the proof transformations consists, from a proof of a sequent using classical additives, to find a proof of the sequent where classical additives are replaced by their second order version. We write $[U]$ the formula obtained by substituting $\oplus'$ for $\oplus$ and $\&'$ for $\&$.

Definition 3 *If U is a formula, $[U]$ is defined by induction by:*

- $[A] = A$ *if A is an atom or a constant.*
- $[U \otimes V] = [U] \otimes [V]$ *and* $[U \wp V] = [U] \wp [V]$ *for multiplicatives.*
- $[U \oplus V] = [U] \oplus' [V] = \forall X.?(X^{\perp} \otimes [U]) \wp ?(X^{\perp} \otimes [V]) \wp X$ *with X not free in U or V.*
- $[U \& V] = [U] \&' [V] = \exists X.!(X^{\perp} \wp [U]) \otimes !(X^{\perp} \wp [V]) \otimes X$ *with X not free in U or V.*
- $[!U] = ![U]$ *et* $[?U] = ?[U]$ *for exponentials.*
- $[\forall X.U] = \forall X.[U]$ *et* $[\exists X.U] = \exists X.[U]$ *for quantifiers.*

We write $[U_1, \ldots, U_n]$ for $[U_1], \ldots, [U_n]$.

Remark: Theorem 2 shows that $\vdash U_1, \ldots, U_n$ is provable if and only if $\vdash [U_1, \ldots, U_n]$ is provable.

Definition 4 *Let $\mathcal{P}$ be a proof of $\vdash U_1, \ldots, U_n$ (written $\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash U_1, \ldots, U_n\end{array}$), $[\mathcal{P}]$ is defined following the last inference of $\mathcal{P}$ by:*

- $$
\left[\dfrac{\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash \Gamma, U\end{array}}{\vdash \Gamma, U \oplus V}\,1\oplus\right]
=
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^{\perp}, X} \quad \left[\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash \Gamma, U\end{array}\right]}{\vdash [\Gamma], X^{\perp} \otimes [U], X}\,\otimes}{\vdash [\Gamma], ?(X^{\perp} \otimes [U]), ?(X^{\perp} \otimes [V]), X}\,?}{\vdash [\Gamma], ?(X^{\perp} \otimes [U]) \wp ?(X^{\perp} \otimes [V]) \wp X}\,\wp}{\vdash [\Gamma], \forall X.?(X^{\perp} \otimes [U]) \wp ?(X^{\perp} \otimes [V]) \wp X}\,\forall X}{\vdash [\Gamma], [U \oplus V]}\,\mathbf{Def}
$$
  where $\mathcal{P}$ is the premisse of the left $\oplus$ rule.
- *The right $\oplus$ rule is similar.*

- $$\left[\dfrac{\begin{matrix}\mathcal{P}_1 & & \mathcal{P}_2\\ \vdots & & \vdots \\ \vdash \Gamma, U & & \vdash\Gamma, V\end{matrix}}{\vdash \Gamma, U \& V}\,\&\right] = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\left[\begin{matrix}\mathcal{P}_1\\ \vdots\\ \vdash\Gamma,U\end{matrix}\right]}{\vdash C,[U]}\wp(\bot)}{\vdash C\wp[U]}\wp}{\vdash !(C\wp[U])}! \quad \dfrac{\dfrac{\dfrac{\left[\begin{matrix}\mathcal{P}_2\\ \vdots\\ \vdash\Gamma,V\end{matrix}\right]}{\vdash C,[V]}\wp(\bot)}{\vdash C\wp[V]}\wp}{\vdash !(C\wp[V])}! \quad \dfrac{}{\vdash [\Gamma], C^\perp}\otimes(1)}{\vdash [\Gamma], !(C\wp[U])\otimes !(C\wp[V])\otimes C^\perp}\otimes}{\vdash [\Gamma], \exists X.!(X^\perp\wp[U])\otimes!(X^\perp\wp[V])\otimes X^\perp}\exists X = C^\perp}{\vdash [\Gamma],[U\&V]}\mathbf{Def}$$ *where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the two premisses of the & rule and where $C = [A_1]\wp\cdots\wp[A_n]$ if $\Gamma = A_1, \ldots, A_n$ ($C = \bot$ if n is null).*

- $$\left[\dfrac{\begin{matrix}\mathcal{P}_1 & & \mathcal{P}_k\\ \vdots & \cdots & \vdots \\ \Gamma_1 & & \Gamma_k\end{matrix}}{\vdash U_1, \ldots, U_n}\,R\right] = \dfrac{\left[\begin{matrix}\mathcal{P}_1\\ \vdots\\ \Gamma_1\end{matrix}\right] \cdots \left[\begin{matrix}\mathcal{P}_k\\ \vdots\\ \Gamma_k\end{matrix}\right]}{\vdash [U_1], \ldots, [U_n]}\,R$$ *for the other inference. $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are les premisses of the rule.*

Theorem 5 *If $\mathcal{P}$ is a proof of $\vdash U_1, \ldots, U_n$ then $[\mathcal{P}]$ is a proof of $\vdash [U_1], \ldots, [U_n]$. Moreover, $[\mathcal{P}]$ is a proof without using the classical additives of linear logic.*

Proof: By induction on $\mathcal{P}$. $\square$

2.3 Emulation of Cut Elimination

Normalization of a proof may be emulated in its transformed proof.

Lemma 6 *If $\mathcal{P}$ is a proof and $\mathcal{P}$ reduces in one step to $\mathcal{P}'$ without using commutative conversion of $\oplus$ and & then $[\mathcal{P}]$ reduces (in one or more steps) to $[\mathcal{P}']$:*

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathbf{Red}"] \arrow[d, "{[\,]}"] & \mathcal{P}' \arrow[d, "{[\,]}"] \\
{[\mathcal{P}]} \arrow[r, "\mathbf{Red}^+"] & {[\mathcal{P}']}
\end{tikzcd}$$

Proof: We only need to prove that the transformed proof corresponding to the two different cases of cut between a $\oplus$ and a & can reduce like the classical additives.

For the cut between the & rule and the left $\oplus$ rule (the other case is similar):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\left[\begin{array}{c}\mathcal{P}_1\\ \vdots\\ \vdash \Gamma, U\end{array}\right]}{\vdash C,[U]}\;\wp(\bot)}{\vdash C\wp[U]}\;\wp}{\vdash !(C\wp[U])}\;!
\qquad
\dfrac{\dfrac{\dfrac{\left[\begin{array}{c}\mathcal{P}_2\\ \vdots\\ \vdash \Gamma, V\end{array}\right]}{\vdash C,[V]}\;\wp(\bot)}{\vdash C\wp[V]}\;\wp}{\vdash !(C\wp[V])}\;!
\qquad
\dfrac{}{\vdash [\Gamma], C^\perp}\;\otimes(1)
}{\vdash [\Gamma], !(C\wp[U])\otimes !(C\wp[V])\otimes C^\perp}\;\otimes
}{\vdash [\Gamma], \exists X.!(X^\perp\wp[U])\otimes !(X^\perp\wp[V])\otimes X^\perp}\;\exists X = C^\perp
}{\vdash [\Gamma],[U\&V]}\;\textbf{Def}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\vdash X^\perp, X}\qquad \left[\begin{array}{c}\mathcal{P}\\ \vdots\\ \vdash U^\perp, \Delta\end{array}\right]}{\vdash X^\perp\otimes[U^\perp], X, [\Delta]}\;\otimes
}{\vdash ?(X^\perp\otimes[U^\perp]), ?(X^\perp\otimes[V^\perp]), X, [\Delta]}\;?
}{\vdash ?(X^\perp\otimes[U^\perp])\wp ?(X^\perp\otimes[V^\perp])\wp X, [\Delta]}\;\wp
}{\vdash \forall X.?(X^\perp\otimes[U^\perp])\wp ?(X^\perp\otimes[V^\perp])\wp X, [\Delta]}\;\forall X
}{\vdash [U^\perp\oplus V^\perp],[\Delta]}\;\textbf{Def}
}{\vdash [\Gamma],[\Delta]}\;\textbf{Cut}
$$

$$\xrightarrow{\mathbf{Red}^+}$$

$$
\dfrac{\left[\begin{array}{c}\mathcal{P}_1\\ \vdots\\ \vdash \Gamma, U\end{array}\right]\qquad \left[\begin{array}{c}\mathcal{P}\\ \vdots\\ \vdash U^\perp, \Delta\end{array}\right]}{\vdash [\Gamma],[\Delta]}\;\textbf{Cut}
$$

$\square$

Theorem 7 *If $\mathcal{P}$ is a proof and $\mathcal{P}$ reduces to $\mathcal{P}'$ without using commutative conversion of $\oplus$ and & then $[\mathcal{P}]$ reduces to $[\mathcal{P}']$:*

$$
\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathbf{Red}^*"] \arrow[d, "{[\,]}"] & \mathcal{P}' \arrow[d, "{[\,]}"] \\
{[\mathcal{P}]} \arrow[r, "\mathbf{Red}^*"] & {[\mathcal{P}']}
\end{tikzcd}
$$

Proof: By induction on the derivation from $\mathcal{P}$ to $\mathcal{P}'$ and using the Lemma 6. $\square$

3 Elimination of Environment in !-intoduction

For exponentials the aims of this section are to introduce *second order* exponentials, to find proof transformations and to show that cut elimination may be emulated in the same scheme as for the additives.

3.1 Second Order Exponentials

In the previous section, the proof transformation of a & rule creats two proofs of $\vdash !(C\wp[V])$ and $\vdash !(C\wp[U])$. As we can see, ! is introduced with an empty environment. This transformation separates completely the context and the two alternatives. The context will be given only when one of the alternatives will be chosen (during cut elimination).

The classical rules for exponentials are:

$$\dfrac{\vdash ?\Gamma, U}{\vdash ?\Gamma, !U}\,! \qquad \dfrac{\vdash \Gamma}{\vdash \Gamma, ?U}\,W? \qquad \dfrac{\vdash \Gamma, ?U, ?U}{\vdash \Gamma, ?U}\,C? \qquad \dfrac{\vdash \Gamma, U}{\vdash \Gamma, ?U}\,D?$$

Thus, as the two different $\oplus$ rules give the two sub-formulas $X^\perp \wp U$ et $X^\perp \wp V$ of $\&'$, the three ? rules ? give three sub-formulas for $!'$.

- The weakening rule corresponds to $X^\perp \wp 1$ which erazes an X.
- The contraction rule corresponds to $X^\perp \wp (X \otimes X)$ which duplicates X.
- The dereliction rule corresponds to $X^\perp \wp U$ which transforms X to U.

Definition 8 *The second order exponentials $?'$ and $!'$ are defined by:*

$$\begin{array}{lcl} ?'U & \stackrel{\mathbf{Def}}{=} & \forall X.?(X^\perp \otimes \bot) \wp ?(X^\perp \otimes (X \wp X)) \wp ?(X^\perp \otimes U) \wp X \\ !'U & \stackrel{\mathbf{Def}}{=} & \exists X.!(X^\perp \wp 1) \otimes !(X^\perp \wp (X \otimes X)) \otimes !(X^\perp \wp U) \otimes X \end{array}$$

Contrary to the additives, those second order exponentials are not equivalent to the classical ones. However, they are equivalents to the properties of weakening and contraction. Thus, they are a weak form of the classical exponentials.

Theorem 9 *Let $D^\perp$ and U be two formulas.*
If $\left\{ \begin{array}{l} \vdash D, 1 \\ \vdash D, D^\perp \otimes D^\perp \\ \vdash D, U \end{array} \right.$ *are provable then $\vdash D, !'U$ is provable.*

Proof: If we have three proofs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ of the sequents $\vdash D, 1$, $\vdash D, D^\perp \otimes D^\perp$ and $\vdash D, U$, we can build a proof of $\vdash D, !'U$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{P}_1\\ \vdots \\ \vdash D, 1\end{array}}{\vdash D \wp 1}\,\wp}{\vdash !(D \wp 1)}\,! \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{P}_2\\ \vdots \\ \vdash D, D^\perp \otimes D^\perp\end{array}}{\vdash D \wp (D^\perp \otimes D^\perp)}\,\wp}{\vdash !(D \wp (D^\perp \otimes D^\perp))}\,! \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{P}_3\\ \vdots \\ \vdash D, U\end{array}}{\vdash D \wp U}\,\wp}{\vdash !(D \wp U)}\,! \quad \dfrac{}{\vdash D, D^\perp}}{\vdash D, !(D \wp 1) \otimes !(D \wp (D^\perp \otimes D^\perp)) \otimes !(D \wp U) \otimes D^\perp}\,\otimes}{\vdash D, \exists X.!(X^\perp \wp 1) \otimes !(X^\perp \wp (X \otimes X)) \otimes !(X^\perp \wp U) \otimes X}\,\exists X = D^\perp}{\vdash D, !'U}\,\mathbf{Def}$$

□

Theorem 10 *If U is a formula then the following sequents are provable:*

$$\left\{ \begin{array}{l} \vdash ?'U, 1 \\ \vdash ?'U, !'U^\perp \otimes !'U^\perp \\ \vdash ?'U, U^\perp \end{array} \right.$$

Proof: Proof $\mathcal{P}_{W?}(U)$ of $\vdash ?'U, 1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash X^{\perp}, X} \quad \overline{\vdash \perp, 1}}{\vdash X^{\perp} \otimes \perp, X, 1}\otimes}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, 1}\,?}{\vdash ?(X^{\perp} \otimes \perp) \wp ?(X^{\perp} \otimes (X \wp X)) \wp ?(X^{\perp} \otimes U) \wp X, 1}\wp}{\vdash \forall X.?(X^{\perp} \otimes \perp) \wp ?(X^{\perp} \otimes (X \wp X)) \wp ?(X^{\perp} \otimes U) \wp X, 1}\forall X}{\vdash ?'U, 1}\,\mathbf{Def}$$

Proof $\mathcal{P}_{C?}(U)$ of $\vdash ?'U, !'U^{\perp} \otimes !'U^{\perp}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{P} \\ \vdots\end{matrix} \quad \begin{matrix}\mathcal{P} \\ \vdots\end{matrix}}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, X, !'U^{\perp} \otimes !'U^{\perp}}\,?,\otimes}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X \wp X, !'U^{\perp} \otimes !'U^{\perp}}\wp \quad \overline{X^{\perp}, X}}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X^{\perp} \otimes (X \wp X), X, !'U^{\perp} \otimes !'U^{\perp}}\otimes}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, !'U^{\perp} \otimes !'U^{\perp}}\,?}{\vdash ?(X^{\perp} \otimes \perp) \wp ?(X^{\perp} \otimes (X \wp X)) \wp ?(X^{\perp} \otimes U) \wp X, !'U^{\perp} \otimes !'U^{\perp}}\wp}{\vdash \forall X.?(X^{\perp} \otimes \perp) \wp ?(X^{\perp} \otimes (X \wp X)) \wp ?(X^{\perp} \otimes U) \wp X, !'U^{\perp} \otimes !'U^{\perp}}\forall X}{\vdash ?'U, !'U^{\perp} \otimes !'U^{\perp}}\,\mathbf{Def}$$

with $\mathcal{P}$ the proof of $\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, !'U^{\perp}$:

$$\dfrac{\dfrac{\dfrac{\overline{\vdash ?(X^{\perp} \otimes \perp), !(X \wp 1)} \quad \overline{\vdash ?(X^{\perp} \otimes (X \wp X)), !(X \wp (X^{\perp} \otimes X^{\perp}))} \quad \overline{\vdash ?(X^{\perp} \otimes U^{\perp}), !(X \wp U^{\perp})} \quad \overline{\vdash X, X^{\perp}}}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, !(X \wp 1) \otimes !(X \wp (X^{\perp} \otimes X^{\perp})) \otimes !(X \wp U^{\perp}) \otimes X^{\perp}}\otimes}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, \exists X.!(X \wp 1) \otimes !(X \wp (X^{\perp} \otimes X^{\perp})) \otimes !(X \wp U^{\perp}) \otimes X^{\perp}}\exists X = X^{\perp}}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, !'U^{\perp}}\,\mathbf{Def}$$

Proof $\mathcal{P}_{D?}(U)$ of $\vdash ?'U, U^{\perp}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash X^{\perp}, X} \quad \overline{\vdash U, U^{\perp}}}{\vdash X^{\perp} \otimes U, X, U^{\perp}}\otimes}{\vdash ?(X^{\perp} \otimes \perp), ?(X^{\perp} \otimes (X \wp X)), ?(X^{\perp} \otimes U), X, U^{\perp}}\,?}{\vdash ?(X^{\perp} \otimes \perp) \wp ?(X^{\perp} \otimes (X \wp X)) \wp ?(X^{\perp} \otimes U) \wp X, U^{\perp}}\wp}{\vdash \forall X.?(X^{\perp} \otimes \perp) \wp ?(X^{\perp} \otimes (X \wp X)) \wp ?(X^{\perp} \otimes U) \wp X, U^{\perp}}\forall X}{\vdash ?'U, U^{\perp}}\,\mathbf{Def}$$

$\square$

We can extend those proofs to a sequence of formulas:

Theorem 11 *If $\Gamma = A_1, \ldots, A_n$ is a sequence of formulas then the following sequents are provable:*

$$\begin{cases} \vdash ?'\Gamma, 1 \\ \vdash ?'\Gamma, C \otimes C \end{cases}$$

where $C = !'A_1^{\perp} \otimes \cdots \otimes !'A_n^{\perp}$ ($\perp$ if $n = 0$).
We write $\mathcal{P}_{W?}(\Gamma)$ the proof of $\vdash ?'\Gamma, 1$ and $\mathcal{P}_{C?}(\Gamma)$ the proof of $\vdash ?'\Gamma, C \otimes C$.

Proof: By induction on n in the same spirit as the proofs of the theorem 10. $\square$
Those proofs, except the substitution of $A_1, \ldots, A_k$ by other formulas depend only on the number of formulas of Γ. They contain no cut.

3.2 Transformations of Exponentials

These second order exponentials allow us to define a transformation on formulas, sequents and finally on proofs.

Definition 12 *If U is a formula, $[U]$ is defined by induction by:*

- $[A] = A$ *if A is an atom or a constant.*
- $[U \otimes V] = [U] \otimes [V]$ *and* $[U \wp V] = [U] \wp [V]$ *for the multiplicatives.*
- $[U \& V] = [U] \& [V]$ *and* $[U \oplus V] = [U] \oplus [V]$ *for the additives.*
- $[!U] = !'[U] = \exists X.!(X^\perp \wp 1) \otimes !(X^\perp \wp (X \otimes X)) \otimes !(X^\perp \wp [U]) \otimes X$ *where X is not free in U.*
- $[?U] = ?'[U] = \forall X.?(X^\perp \otimes \bot) \wp ?(X^\perp \otimes (X \wp X)) \wp ?(X^\perp \otimes [U]) \wp X$ *where X is not free in U.*
- $[\forall X.U] = \forall X.[U]$ *et* $[\exists X.U] = \exists X.[U]$ *for the quantifiers.*

We write $[U_1, \ldots, U_n]$ for $[U_1], \ldots, [U_n]$.

Now, we can introduce proof transformations such that if $\mathcal{P}$ is a proof of $\vdash U_1, \ldots, U_n$ then $[\mathcal{P}]$ is a proof of $\vdash [U_1, \ldots, U_n]$.

Definition 13 *If $\mathcal{P}$ is a proof of a $\vdash U_1, \ldots, U_n$ (written $\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash U_1, \ldots, U_n\end{array}$), $[\mathcal{P}]$ is defined following the main rule of $\mathcal{P}$:*

- $$\left[\dfrac{\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?U}\,W?\right] = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X} \qquad \dfrac{\left[\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash \Gamma\end{array}\right]}{\vdash [\Gamma], \bot}\,\bot}{\vdash [\Gamma], X^\perp \otimes \bot, X}\,\otimes}{\vdash [\Gamma], ?(X^\perp \otimes \bot), ?(X^\perp \otimes (X \wp X)), ?(X^\perp \otimes [U]), X}\,?}{\vdash [\Gamma], ?(X^\perp \otimes \bot) \wp ?(X^\perp \otimes (X \wp X)) \wp ?(X^\perp \otimes [U]) \wp X}\,\wp}{\vdash [\Gamma], \forall X.?(X^\perp \otimes \bot) \wp ?(X^\perp \otimes (X \wp X)) \wp ?(X^\perp \otimes [U]) \wp X}\,\forall X}{\vdash [\Gamma], [?U]}\,\mathbf{Def}$$ *where $\mathcal{P}$ is the premise of the weakening rule.*

- $$\left[\dfrac{\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash \Gamma, ?U, ?U\end{array}}{\vdash \Gamma, ?U}\,C?\right] = \dfrac{\dfrac{\left[\begin{array}{c}\mathcal{P}\\ \vdots \\ \vdash \Gamma, ?U, ?U\end{array}\right]}{\vdash [\Gamma], [?U] \wp [?U]}\,\wp \qquad \begin{array}{c}\mathcal{P}_{C?}([U])\\ \vdots \\ \vdash [!U^\perp] \otimes [!U^\perp], [?U]\end{array}}{\vdash [\Gamma], [?U]}\,\mathbf{Cut}$$ *where $\mathcal{P}$ is the premise of the contraction rule.*

- $$\left[\dfrac{\begin{matrix}\mathcal{P}\\ \vdots\\ \vdash \Gamma, U\end{matrix}}{\vdash \Gamma, ?U}\,D?\right] = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^\perp, X} \qquad \left[\begin{matrix}\mathcal{P}\\ \vdots\\ \vdash \Gamma, U\end{matrix}\right]}{\vdash [\Gamma], X^\perp \otimes [U], X}\,\otimes}{\vdash [\Gamma], ?(X^\perp \otimes \bot), ?(X^\perp \otimes (X \wp X)), ?(X^\perp \otimes [U]), X}\,?}{\vdash [\Gamma], ?(X^\perp \otimes \bot) \wp ?(X^\perp \otimes (X \wp X)) \wp ?(X^\perp \otimes [U]) \wp X}\,\wp}{\vdash [\Gamma], \forall X.?(X^\perp \otimes \bot) \wp ?(X^\perp \otimes (X \wp X)) \wp ?(X^\perp \otimes [U]) \wp X}\,\forall X}{\vdash [\Gamma], [?U]}\,\mathbf{Def}$$ *where $\mathcal{P}$ is the premise of the dereliction rule.*

- $$\left[\dfrac{\begin{matrix}\mathcal{P}\\ \vdots\\ \vdash ?\Gamma, U\end{matrix}}{\vdash ?\Gamma, !U}\,!\right] = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{P}_{W?}([\Gamma])\\ \vdots\\ \vdash [?\Gamma], 1\end{matrix}}{\vdash C, 1}\,\wp(\bot)}{\vdash C \wp 1}\,\wp}{\vdash !(C \wp 1)}\,! \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{P}_{C?}([\Gamma])\\ \vdots\\ \vdash [?\Gamma], C^\perp \otimes C^\perp\end{matrix}}{\vdash C, C^\perp \otimes C^\perp}\,\wp(\bot)}{\vdash C \wp (C^\perp \otimes C^\perp)}\,\wp}{\vdash !(C \wp (C^\perp \otimes C^\perp))}\,! \quad \dfrac{\dfrac{\dfrac{\left[\begin{matrix}\mathcal{P}\\ \vdots\\ \vdash [?\Gamma], [U]\end{matrix}\right]}{\vdash C, [U]}\,\wp(\bot)}{\vdash C \wp [U]}\,\wp}{\vdash !(C \wp [U])}\,! \quad \dfrac{}{\vdash [?\Gamma], C^\perp}\,\otimes(1)}{\vdash [?\Gamma], !(C \wp 1) \otimes !(C \wp (C^\perp \otimes C^\perp)) \otimes !(C \wp [U]) \otimes C^\perp}\,\otimes}{\vdash [?\Gamma], \exists X.!(X^\perp \wp 1) \otimes !(X^\perp \wp (X \otimes X)) \otimes !(X^\perp \wp [U]) \otimes X}\,\exists X = C^\perp}{\vdash [?\Gamma], [!U]}\,\mathbf{Def}$$ *where $\mathcal{P}$ is the premise of the ! rule and where $C = [A_1] \wp \cdots \wp [A_n]$ if $\Gamma = A_1, \ldots, A_n$ ($C = \bot$ if n=0).*

- $$\left[\dfrac{\begin{matrix}\mathcal{P}_1 & & \mathcal{P}_k\\ \vdots & \ldots & \vdots\\ \Gamma_1 & & \Gamma_k\end{matrix}}{\vdash U_1, \ldots, U_n}\,R\right] = \dfrac{\left[\begin{matrix}\mathcal{P}_1\\ \vdots\\ \Gamma_1\end{matrix}\right] \;\ldots\; \left[\begin{matrix}\mathcal{P}_k\\ \vdots\\ \Gamma_k\end{matrix}\right]}{\vdash [U_1], \ldots, [U_n]}\,R$$ *for the other rules. $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are the premises of the rule.*

Theorem 14 *If $\mathcal{P}$ is a proof of $\vdash U_1, \ldots, U_n$ then $[\mathcal{P}]$ is a proof of $\vdash [U_1], \ldots, [U_n]$. Moreover, ! rule only appears with empty environments in $[\mathcal{P}]$.*

Proof: By induction on $\mathcal{P}$. □

3.3 Emulation of Cut Elimination

Like additives, normalization of a proof may be emulated in its transformed proof.

Lemma 15 *If $\mathcal{P}$ is a proof and $\mathcal{P}$ reduces in one step to $\mathcal{P}'$ without using commutative conversion for exponentials then $[\mathcal{P}]$ reduce (in one or more step) to $[\mathcal{P}']$.*

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathbf{Red}"] \arrow[d, "{[\,]}"] & \mathcal{P}' \arrow[d, "{[\,]}"] \\
{[\mathcal{P}]} \arrow[r, "\mathbf{Red}^+"] & {[\mathcal{P}']}
\end{tikzcd}$$

Proof: As with additives, we only need to prove that the three different kinds of cut between $!'U$ and $?'U^{\perp}$ reduce as the classical exponentials do. For instance, with a cut between a ! rule and a dereliction:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{P}_{W?}([\Gamma)]\\ \vdash [?\Gamma], 1\end{array}}{\vdash C^{\perp}, 1}\,\wp(\perp)}{\vdash C^{\perp}\wp 1}\,\wp}{\vdash !(C^{\perp}\wp 1)}\,!
\quad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{P}_{C^{\perp}?}([\Gamma)]\\ \vdash [?\Gamma], C\otimes C\end{array}}{\vdash C^{\perp}, C\otimes C}\,\wp(\perp)}{\vdash C^{\perp}\wp(C\otimes C)}\,\wp}{\vdash !(C^{\perp}\wp(C\otimes C))}\,!
\quad
\dfrac{\dfrac{\dfrac{\left[\begin{array}{c}\mathcal{P}\\ \vdash [?\Gamma], [U]\end{array}\right]}{\vdash C^{\perp}, [U]}\,\wp(\perp)}{\vdash C^{\perp}\wp[U]}\,\wp}{\vdash !(C^{\perp}\wp[U])}\,!
\quad
\dfrac{}{\vdash [?\Gamma], C}\,\otimes(1)
}{\vdash [?\Gamma], !(C^{\perp}\wp 1)\otimes !(C^{\perp}\wp(C\otimes C))\otimes !(C^{\perp}\wp[U])\otimes C}\,\otimes
}{\vdash [?\Gamma], \exists X.!(X^{\perp}\wp 1)\otimes !(X^{\perp}\wp(X\otimes X))\otimes !(X^{\perp}\wp[U])\otimes X}\,\exists X = C
}{\vdash [?\Gamma], [!U]}\,\mathbf{Def}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash X^{\perp}, X}\quad \left[\begin{array}{c}\mathcal{P}\\ \vdash U^{\perp}, \Delta\end{array}\right]}{\vdash X^{\perp}\otimes [U^{\perp}], X, [\Delta]}\,\otimes}{\vdash ?(X^{\perp}\otimes\perp), ?(X^{\perp}\otimes(X\wp X)), ?(X^{\perp}\otimes[U^{\perp}]), X, [\Delta]}\,?}{\vdash ?(X^{\perp}\otimes\perp)\wp ?(X^{\perp}\otimes(X\wp X))\wp ?(X^{\perp}\otimes[U^{\perp}])\wp X, [\Delta]}\,\wp}{\vdash \forall X.?(X^{\perp}\otimes\perp)\wp ?(X^{\perp}\otimes(X\wp X))\wp ?(X^{\perp}\otimes[U^{\perp}])\wp X, [\Delta]}\,\forall X}{\vdash [?U^{\perp}], [\Delta]}\,\mathbf{Def}
}{\vdash [\Gamma], [\Delta]}\,\mathbf{Cut}
$$

$$\xrightarrow{\mathbf{Red}^{+}}$$

$$
\dfrac{\left[\begin{array}{c}\mathcal{P}_1\\ \vdash \Gamma, U\end{array}\right]\quad\left[\begin{array}{c}\mathcal{P}\\ \vdash U^{\perp}, \Delta\end{array}\right]}{\vdash [\Gamma], [\Delta]}\,\mathbf{Cut}
$$

$\square$

Theorem 16 *If $\mathcal{P}$ reduces to $\mathcal{P}$ without using commutative conversion of exponentials then $[\mathcal{P}]$ reduces to $[\mathcal{P}']$.*

$$
\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathbf{Red}^{*}"] \arrow[d, "{[\,]}"] & \mathcal{P}' \arrow[d, "{[\,]}"] \\
{[\mathcal{P}]} \arrow[r, "\mathbf{Red}^{*}"] & {[\mathcal{P}']}
\end{tikzcd}
$$

Proof: By induction on the derivation from $\mathcal{P}$ to $\mathcal{P}'$ and by using Lemma 15 $\square$

4 Constant Elimination

The second order quantifiers allow us to eliminate the multiplicative and additive constants of linear logic in the same spirit as the previous transformations.

Definition 17 *The second order constants are defined by:*

$$
\begin{array}{lcl}
1' & \stackrel{\mathbf{Def}}{=} & \forall X.X^{\perp}\wp X \\
\perp' & \stackrel{\mathbf{Def}}{=} & \exists X.X^{\perp}\otimes X \\
0' & \stackrel{\mathbf{Def}}{=} & \forall X.X \\
\top' & \stackrel{\mathbf{Def}}{=} & \exists X.X
\end{array}
$$

The new constants are linearly equivalent to the classical ones. They behave like classic ones as respects to cut elimination (without commutative conversion of $\perp$ and $\top$).

5 Conclusion

We can apply the different transformations all together on formulas, sequents and proofs. The result is linear logic proofs without any constant and additive and where ! rules have empty environments. Normalization (without commutative conversion) of a proof can be emulated in the transformed one.

This mechanism may be compared to different codings of linear logic proofs (and λ-calculus) into reduction net systems [Laf95]. All those translations emulate cut elimination without commutative conversion. Our linear logic proof transformations act similarly for cut elimination but take place inside linear logic. The strategies used by those systems are globally the same as ours. Thus, the proof transformations give a theoretical justification for those codings: they can be obtained first by transforming a proof then coding it in the object language. This decomposition divides those emulations in two mechanisms. The first one (our proof transformations) inhibits commutative conversions and the second one uses a simple translation (because proof constructions are less numerous in transformed proofs).

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