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Comparative Analysis and Synthesis of Plane and Spherical Four-Hinge Mechanisms

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Abstract—The conditions of motion transmission and pressure angles in plane and spherical mechanisms are considered. Algorithms for the determination of the indicated angles are developed, and a comparison of the conditions of motion transmission in these mechanisms is carried out.

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Pressure angles in spherical mechanisms, in contrast to the ones in plane linkage mechanisms, have tangential and normal components [1–3]. The former defines the condition of force flow within the motion plane of an output element and is equivalent to the pressure angle in a plane mechanism. The normal component defines the axial component of the response to the output element and affects the friction torque in the revolute pair output element–pedestal. A comparative analysis of spherical and plane mechanisms can be carried out using kinematic parameters, which characterize the motion law of the output element, and using the pressure angle, which characterizes the flow of force.

A spherical four-hinged mechanism is defined by four interaxial angles α_1, α_2, α_3, and α_4 of the revolute pairs of its elements and the coefficient $\mu = OB/OA$, which defines the relative position of the A and B centers of the hinges of the connecting rod and represents the relationship between the radii of the spheres, which include the above-indicated centers. As a spherical mechanism is a specific case of a spatial *BCCB* mechanism, in order to determine the kinematic parameters and pressure angle, we will use formulas for spatial mechanisms [1, 2], taking into account the fact that the axes of all revolute pairs of spherical mechanisms intersect at one point (Fig. 1).

The normal component of pressure angles in spherical mechanisms is determined using the formula [3]

$$\sin\nu_{n12} = (\mu\cos\alpha_3 - \cos\alpha_1\cos\alpha_4 \pm \sin\alpha_1\sin\alpha_4)/\sqrt{1+\mu^2-2\mu\cos\alpha_2},$$

where subscripts 1 and 2 correspond to the maximum and minimum pressure angle values.

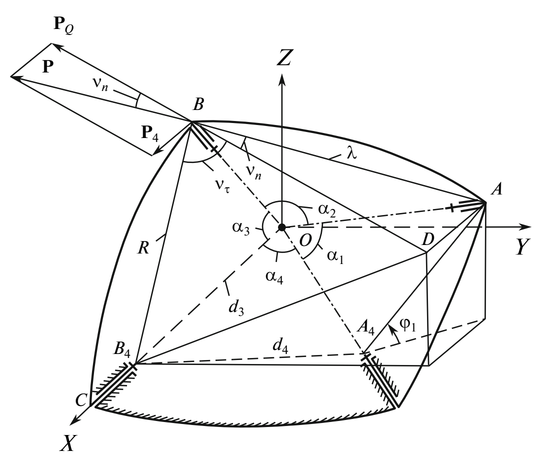

Fig. 1. Components of pressure angles of spherical four-hinge mechanisms.

The extreme values of the tangential component of a pressure angle are determined using the formulas [3]

$$\sin\nu_{\tau 12} = \frac{\mu\sin^2\alpha_3 - \cos\alpha_2 + \cos\alpha_3\cos(\alpha_4 \pm \alpha_1)}{\sin\alpha_3\sqrt{1+\mu^2 - 2\mu\cos\alpha_2 - [\mu\cos\alpha_3 - \cos(\alpha_4 \pm \alpha_1)]}}, \quad (1)$$

$$\sin\nu_{\tau 34} = \frac{a + b(k_3/k_1)}{2R\sqrt{\lambda^2 - (m - n(k_3/k_1))^2}}, \quad (2)$$

where the following designations are accepted for $AA_4 = 1$ and $OA/OB = \mu$:

$$a = R^2 + \lambda^2 - 1 - d_4, \quad R = \mu\sin\alpha_3/\sin\alpha_1, \quad \lambda = \sqrt{1+\mu^2 - 2\mu\cos\alpha_2}/\sin\alpha_1,$$

$$d_4 = d_1^2 + d_3^2 - 2d_1d_3\cos\alpha_4, \quad b = 2d_3\sin\alpha_4, \quad m = d_3 - d_1\cos\alpha_4,$$

$$n = \sin\alpha_4, \quad k_1 = -n(mb + an), \quad k_3 = (\lambda^2 - m^2)b - amn;$$

λ and R represent intervals AB and BB_4.

The $\nu_{\tau 3}$ and $\nu_{\tau 4}$ extreme values, which are determined using (2), only take place when $k_3 \pm k_1 \geq 0$ [3]; otherwise, ν_τ only has two extreme values $\nu_{\tau 1} = \min\nu_\tau$ and $\nu_{\tau 2} = \max\nu_\tau$, which are determined using (1).

The extreme condition $\nu'_\tau = 0$ of the tangential component of a pressure angle is transformed in the spherical mechanism as follows: $f(\varphi_1) = (k_1\cos\varphi_1 + k_3)\sin\varphi_1 = 0$, which is used to determine the $\nu_{\tau 34}$ extreme values (in the case of $k_3 \pm k_1 \geq 0$).

In order to provide the best motion transmission for both spherical and plane mechanisms, it is necessary to fulfill the conditions [4] $\min\nu_\tau + \max\nu_\tau = 0$ or $\max\gamma_\tau + \min\gamma_\tau = \pi$, where γ_τ is the transmission angle.

In the case of two ν_τ extreme values, the considered conditions are transformed into the quadratic equation $p_2\mu^2 + p_1\mu + p_4 = 0$, where the p_2, p_1, and p_4 coefficients are determined using the formulas

$$p_2 = \sin^2\alpha_3(\cos\alpha_1\cos\alpha_4 - \cos\alpha_2\cos\alpha_3),$$

$$p_1 = \cos\alpha_3(\cos^2\alpha_1 + \cos^2\alpha_2 - \cos^2\alpha_3 + \cos^2\alpha_4) - 2\cos\alpha_1\cos\alpha_2\cos\alpha_4,$$

$$p_4 = (\cos\alpha_1\cos\alpha_3 - \cos\alpha_2\cos\alpha_4)(\cos\alpha_3\cos\alpha_4 - \cos\alpha_1\cos\alpha_2).$$

The following interaxial angles are selected for the family of spherical crank-rocker mechanisms: $\alpha_4 = 90°$, $\alpha_3 = 80°$, $\alpha_1 = 15°–25°$, and $\alpha_2 = 60°–80°$ ($\Delta\alpha_1 = 5°$, $\Delta\alpha_2 = 10°$).

In order to compare them with plane crank-rocker mechanisms, the following geometric parameters are selected: $L_1 = 1$, $L_2 = 5$, $L_4 = 3.5–4.5$ ($\Delta L_4 = 0.5$).

The movements of the mechanism's output element are characterized by the dimensionless coefficients of its velocity (K_ω), acceleration (K_ε), and power (K_p), the values of which are determined using the formulas [2, 5]

$$K_\omega = \varphi'_3\varphi_\Psi/\Psi, \quad K_\varepsilon = \varphi''_3\varphi_\Psi^2/\Psi, \quad K_p = K_\omega K_\varepsilon, \quad (3)$$

where Ψ is the range of the rotation angle of the output element (rocker swing angle), the value of which is determined using the formula $\Psi = |\varphi_3(\varphi_{10}) - \varphi_3(\varphi_{12})|$; $\varphi_3(\varphi_{12})$ and $\varphi_3(\varphi_{10})$ are the maximum and minimum values of angle φ_3; φ_Ψ is the rotation angle of the crank, corresponding to angle Ψ, the value of which is determined by the difference in the values of the φ_1 input angle in extreme positions indicated by subscripts 0 and 2: $\varphi_\Psi = |\varphi_{12} - \varphi_{10}|$.

The maximum values of the dimensionless coefficients are determined using the formulas

$$\max K_\omega = \max|\varphi'_3|\varphi_\Psi/\Psi, \quad \max K_\varepsilon = \max|\varphi''_3|\varphi_\Psi^2/\Psi, \quad \max K_p = \max|K_\omega K_\varepsilon|. \quad (4)$$

The moment of resistance M_3 at the output element and the moment of resistance M_r adjusted to the input element are connected by the relationship

$$M_r = M_3\left(\frac{\omega_3}{\omega_1}\right) = M_3\varphi'_3. \quad (5)$$

It follows from expressions (3), (4), and (5) that a decrease in $\max K_\omega$ ceteris paribus (M_3, φ_Ψ, Ψ) results in a decrease in the adjusted moment of resistance M_r. Therefore, the dimensionless K_ω coefficient, which is a kinematic parameter, is the mechanism's power characteristic at the same time.

The M^i moment of inertial forces and moments of the mechanism's elements adjusted to its input element is determined by the sum of all elementary works. This moment is the moment of resistance for the engine at the phase of accelerated motion of the mechanism's elements, while it is transmitted to the engine at the phase of decelerated motion of the mechanism's elements.

The moment of inertial forces of the output element is determined using the formula

$$M^i = M_3^i\left(\frac{\omega_3}{\omega_1}\right) = \omega_1^2 J_3 \varphi_3' \varphi_3'', \quad (6)$$

where J_3 is the moment of inertia of the output element with respect to its rotation axis. In this case, the power is determined using the formula

$$P^i = M^i\omega_1 = M_3^i\omega_3 = J_3\varepsilon_3\omega_3 = J_3\varphi_3'\varphi_3''\omega_1^3, \quad (7)$$

where M^i is the moment, which is to be exerted by the engine in order to overcome the moment of inertial forces of the M_3^i output element. M^i has an alternating sign, and its average value per cycle is zero. It is positive when the output element moves with acceleration and negative when it moves with deceleration.

Having the values of analogs φ_3' and φ_3'' expressed in terms of the values of the K_ω, K_ε, and K_p dimensionless coefficients determined using (3), we transform dependences (6) and (7) as follows:

$$M^i = (J_3\omega_1^2\Psi^2/\varphi_\Psi^3)K_p, \quad P^i = (J_3\omega_1^3\Psi^2/\varphi_\Psi^3)K_p.$$

It follows from the above-presented that the dimensionless coefficient $K_p = K_\omega K_\varepsilon$, which is a kinematic parameter, characterizes the moment at the input element or power at the same time. When we determine the rocker swing angle of the spherical mechanism Ψ_S sand, using the formula [1]

$$\Psi_p = \arccos\left[\frac{(L_2 - L_1)^2 - (L_4^2 + L_3^2)}{2L_4L_3}\right] - \arccos\left[\frac{(L_1 + L_2)^2 - (L_4^2 + L_3^2)}{2L_4L_3}\right]$$

to determine the rocker swing angle of the plane mechanism, we may find the plane mechanism, which has an equivalent swing angle.

The output element of a two-crank mechanism makes a complete revolution $\Psi = 2\pi$, when a revolution of the input element is complete $\varphi_\Psi = 2\pi$. We obtain the following simplified formulas for K_ω and K_ε: $K_\omega = \varphi_3'$, $K_\varepsilon = 2\pi\varphi_3''$.

The δ coefficient of nonuniformity of motion of the output element for spherical and plane two-crank mechanisms is determined using the formula [1, 6] $\delta = (\omega_{3\max} - \omega_{3\min})/\omega_{3m}$, which transforms into $\delta = \max\varphi_3' - \min\varphi_3'$ after substituting the average value $\omega_{3m} = \omega_1$ of angular velocity ω_3 of the output element.

The extreme values of the transmission angle for plane linkage crank-rocker and two-crank mechanisms are determined as follows (Fig. 2):

$$\gamma_1 = \min\gamma = \arccos\left(\frac{L_2^2 + L_3^2 - (L_4 - L_1)^2}{2L_2L_3}\right), \quad \gamma_2 = \max\gamma = \arccos\left(\frac{L_2^2 + L_3^2 - (L_1 + L_4)^2}{2L_2L_3}\right).$$

We consider Reinsberg's mechanisms [4], which provide the best conditions for flow of force. The following is fulfilled for them: $\gamma_1 + \gamma_2 = 180°$. This condition only takes place when extreme angles γ_1 and γ_2 are symmetrical with respect to the value $\gamma = 90°$, which takes place at $\varphi = 90°$. This position of the mechanism is indicated by dashed lines in Fig. 3. The *C*, *O*, *A*, and *B* points are placed in this position at the circle with an *AC* diameter.

Therefore, the lengths $L_2 = AB$ and $L_3 = BC$ may be changed for the mechanisms under study with the corresponding changes being made in the position of the *B* hinge at the circle and fulfillment of the condition of existence of two cranks.

When we consider the *OABC* mechanism (Fig. 3), we arrive at the condition $L_2^2 + L_3^2 = L_4^2 + L_1^2$, which is applied for the determination of the length of the output crank $L_3 = \sqrt{L_4^2 + L_1^2 - L_2^2}$.

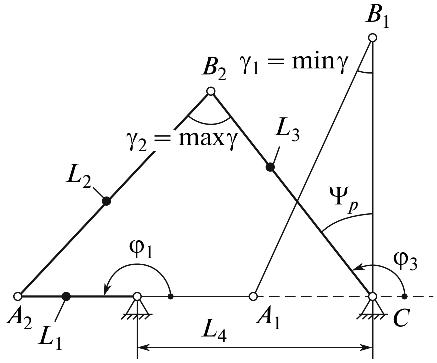

Fig. 2. Extreme transmission angles of crank-rocker mechanisms.

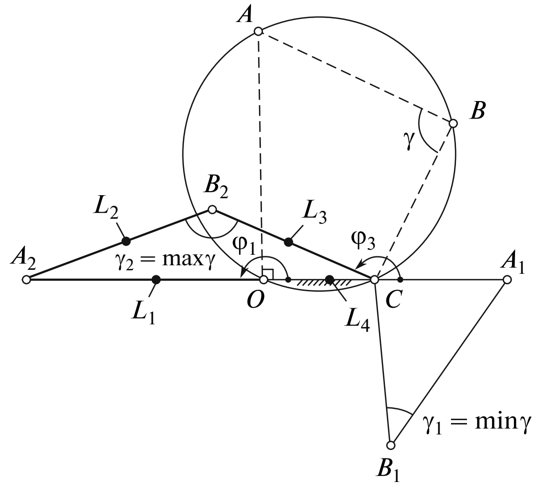

Fig. 3. Extreme transmission angles of two-crank four-hinge mechanisms.

The maximum values of the considered dimensionless coefficients for spherical and plane four-hinge crank-rocker mechanisms at various rocker swing angles obtained using formulas (4) are presented in Table 1.

The extreme values of the pressure angle and its components for spherical and plane crank-rocker mechanisms are presented in Table 2. Their comparison shows that the tangential component of the pressure angle in spherical mechanisms has much higher values.

A graph showing changes in the $\gamma(\varphi_1)$ transmission angle, depending on the input angle φ_1, for mechanisms with the rocker swing angle $\Psi_p = \Psi_s = 30°$ is presented in Fig. 4.

Table 1

Spherical mechanism				Plane mechanism				
$\alpha_3 = 80°$, $\alpha_4 = 90°$				$L_1 = 1$, $L_2 = 5$				
$\alpha_1 = 15°$, $\Psi_s = 30°$				$\Psi_p = 30°$				
α_2, deg	$\max\lvert K_\omega\rvert$	$\max\lvert K_\varepsilon\rvert$	$\max\lvert K_p\rvert$	L_4	L_3	$\max\lvert K_\omega\rvert$	$\max\lvert K_\varepsilon\rvert$	$\max\lvert K_p\rvert$
60	1.71	5.65	5.11	3.5	6.02	1.79	5.14	6.06
70	1.66	5.29	4.58	4	5.24	1.74	5.58	5.95
80	1.63	4.98	4.27	4.5	4.53	1.71	5.88	5.84
$\alpha_1 = 20°$, $\Psi_s = 40°$				$\Psi_p = 40°$				
60	1.79	5.80	5.70	3.5	4.11	1.78	6.21	6.74
70	1.71	5.30	5.03	4	3.59	1.74	6.40	6.52
80	1.67	4.90	4.50	4.5	3.22	1.70	6.46	6.19
$\alpha_1 = 25°$, $\Psi_s = 50°$				$\Psi_p = 50°$				
60	1.90	5.92	6.81	3.5	3.33	1.80	7.14	7.73
70	1.78	5.28	5.61	4	2.9	1.75	7.13	7.24
80	1.72	4.82	5.08	4.5	2.57	1.70	7.00	6.63
α_2	$\max\lvert K_\omega\rvert$	$\max\lvert K_\varepsilon\rvert$	$\max\lvert K_p\rvert$	L_4	L_3	$\max\lvert K_\omega\rvert$	$\max\lvert K_\varepsilon\rvert$	$\max\lvert K_p\rvert$

Table 2

Spherical mechanism						Plane mechanism			
$\alpha_3 = 80°$, $\alpha_4 = 90°$						$L_1 = 1$, $L_2 = 5$			
$\alpha_1 = 15°$, $\Psi_s = 30°$						$\Psi_p = 30°$			
α_2, deg	μ	$\max \nu_n$	$\min \nu_n$	$\max \nu_\tau$	$\min \nu_\tau$	L_4	L_3	$\max \nu$	$\min \nu$
60	0.517	23.73	−11.25	3.18	−3.19	3.5	6.02	65.99	42.91
70	0.354	19.29	−12.12	2.93	−2.87	4	5.24	56.03	31.60
80	0.179	17.12	−13.37	2.72	−2.78	4.5	4.53	47.26	19.70
$\alpha_1 = 20°$, $\Psi_s = 40°$									
60	0.519	29.93	−16.91	4.41	−4.36	3.5	4.11	60.13	31.77
70	0.354	25.43	−17.36	3.94	−3.99	4	3.59	53.58	21.04
80	0.18	22.27	−18.39	3.76	−3.74	4.5	3.22	45.89	9.145
$\alpha_1 = 25°$, $\Psi_s = 50°$									
60	0.521	36.32	−22.55	5.73	−5.68	3.5	3.33	63.64	28.40
70	0.355	31.02	−22.59	5.08	−5.14	4	2.9	57.32	16.86
80	0.18	27.44	−23.42	4.79	−4.83	4.5	2.57	48.86	3.02

The following interaxial angles are selected for spherical mechanisms: $\alpha_1 = 80°$, $\alpha_4 = 10°$, $\alpha_2 = 65°–80°$, $\alpha_3 = 50°–60°$ ($\Delta\alpha_3 = \Delta\alpha_2 = 5°$); the conditions for plane mechanisms are as follows: $L_4 = 1$, $L_1 = 4–6$, and $L_2 = 2–3$ ($\Delta L_1 = \Delta L_2 = 0.5$).

The dependences of the dimensionless coefficients on geometric parameters of mechanisms may be determined based on the presented relationships:

for plane mechanisms,

$$\max|K_\omega| = (aL_2^3 + b)L_1^{-1} + cL_2^2 + d, \quad \max|K_\varepsilon| = (aL_2^3 + b)L_1^{-1} + cL_2^3 + d,$$
$$\max|K_p| = (aL_2^4 + b)L_1^{-1} + cL_2^3 + d; \tag{8}$$

for spherical mechanisms,

$$\max|K_\omega| = (a\alpha_2^{-2} + b)\alpha_3 + c\alpha_2^{-3} + d, \quad \max|K_\varepsilon| = (a\alpha_2^{-6} + b)\alpha_3^{-2} + c\alpha_2^{-7} + d,$$
$$\max|K_p| = (aa_2^{-4} + b)\alpha_3^{-2} + c\alpha_2^{-4} + d. \tag{9}$$

Here, a, b, c, and d are the coefficients determined on the basis of analysis of mechanisms. Dependences (8) and (9) will be further used in the synthesis of mechanisms.

The main parameters of spherical and plane mechanisms, which included the maximum values of the dimensionless coefficients and the δ coefficient of nonuniformity of motion of the output crank, are presented in Tables 3 and 4.

Example. Let us designate spherical two-crank mechanism, which has the following input synthesis parameters: $\alpha_1 = 80°$, $\alpha_4 = 10°$, $\max|K_\varepsilon| = 1.47$, and $\max|K_p| = 1.49$.

When we solve nonlinear equations (9) with respect to the unknown α_2 and α_3 angles, we obtain the required values of the interaxial angles of the mechanism being designed: $\alpha_2 = 66.25°$ and $\alpha_3 = 58.84°$.The obtained mechanism provides the values $\max|K_\varepsilon| = 1.488$ and $\max|K_p| = 1.515$, the deviations of which from the required values do not exceed 1.65%.

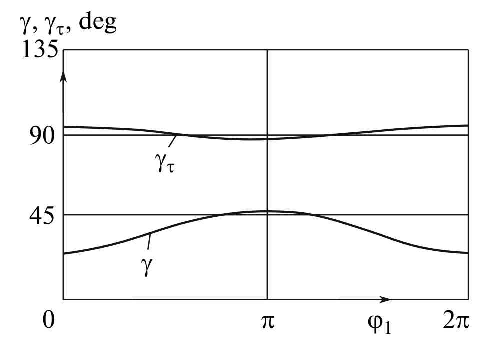

Fig. 4. Changes in transmission angles in plane γ and spherical γ_τ crank-rocker mechanisms.

Table 3

$\alpha_1 = 80°, \quad \alpha_4 = 10°$								
	$\alpha_2 = 65°$				$\alpha_2 = 70°$			
α_3, deg	max|K_ω|	max|K_ε|	max|K_p|	δ	max|K_ω|	max|K_ε|	max|K_p|	δ
50	1.27	2.97	3.11	0.54	1.22	2.09	2.15	0.44
55	1.20	1.94	2	0.41	1.17	1.58	1.612	0.36
60	1.16	1.47	1.49	0.32	1.14	1.26	1.279	0.29
	$\alpha_2 = 75°$				$\alpha_2 = 80°$			
50	1.19	1.71	1.741	0.39	1.17	1.49	1.505	0.36
55	1.16	1.37	1.386	0.32	1.14	1.23	1.238	0.30
60	1.13	1.13	1.137	0.27	1.12	1.03	1.036	0.25

Table 4

$L_4 = 1, \quad L_3 = \sqrt{L_4^2 + L_1^2 - L_2^2}$						
$L_2 = 2$						
L_1	L_3	max|K_ω|	max|K_ε|	max|K_p|	δ	$\nu°$
4	3.61	1.33	2.84	3.166	0.63	33.69
4.5	4.15	1.29	2.42	2.64	0.54	32.8
5	4.69	1.25	2.11	2.271	0.48	32.21
5.5	5.22	1.22	1.88	1.994	0.43	31.79
6	5.74	1.2	1.69	1.783	0.39	31.48
$L_2 = 2.5$						
4	3.28	1.35	2.91	3.234	0.66	29.21
4.5	3.87	1.29	2.41	2.614	0.56	27.69
5	4.44	1.25	2.07	2.213	0.49	26.75
5.5	5	1.22	1.82	1.919	0.43	26.1
6	5.54	1.2	1.63	1.708	0.39	25.64
$L_2 = 3$						
4	2.82	1.39	3.27	3.698	0.75	28.13
4.5	3.5	1.31	2.55	2.78	0.61	25.38
5	4.12	1.26	2.13	2.273	0.51	23.84
5.5	4.78	1.23	1.84	1.923	0.45	22.87
6	5.29	1.2	1.63	1.694	0.4	22.29

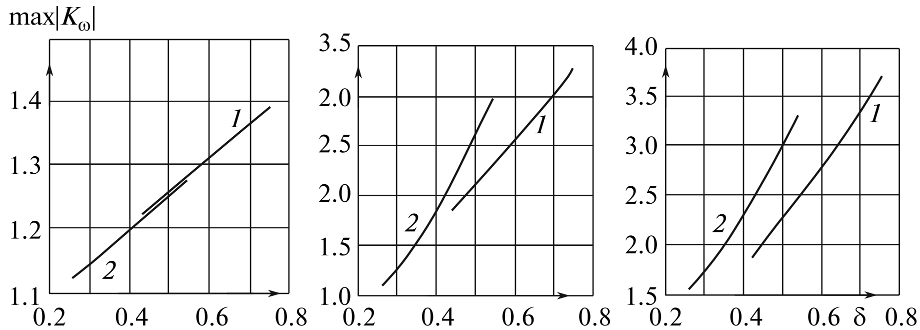

Fig. 5. Changes in the values of the dimensionless coefficients of (a) angular velocity, (b) accelerations, (c) and power.

Graphs showing changes in the maximum values of the dimensionless coefficients of two-crank spherical (*2*) and plane (*1*) mechanisms, depending on the δ coefficient of nonuniformity of motion of the output crank are presented in Fig. 5.

CONCLUSION

Spherical mechanisms are characterized by lower values of inertial forces and engines' moments than plane mechanisms. In addition, spherical mechanisms have better pressure angle values.

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